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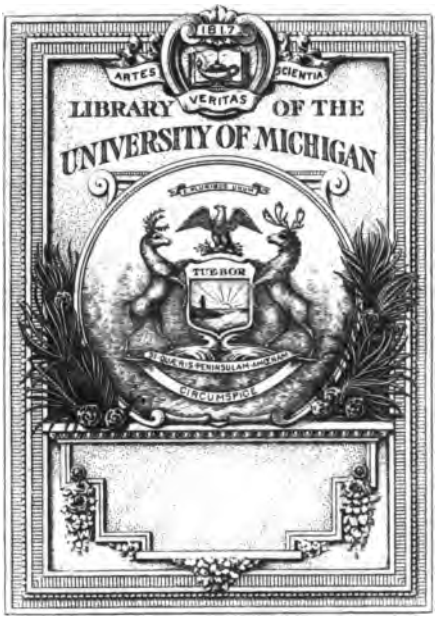
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THE
ELEMENTS
OF
EUCLID,
VIZ.
THE FIRST SIX BOOKS,
WITH
THE ELEVENTH AND TWELFTH.

IN WHICH THE CORRECTIONS OF DR SIMSON ARE GENERALLY ADOPTED, BUT THE ERRORS OVERLOOKED BY HIM ARE CORRECTED, AND THE OBSCURITIES OF HIS AND OTHER EDITIONS EXPLAINED.

ALSO,

SOME OF EUCLID'S DEMONSTRATIONS ARE RESTORED, OTHERS MADE SHORTER AND MORE GENERAL, AND SEVERAL USEFUL PROPOSITIONS ARE ADDED.

TOGETHER WITH
ELEMENTS
OF
PLANE AND SPHERICAL TRIGONOMETRY,
AND
A TREATISE ON PRACTICAL GEOMETRY.

BY *ALEXANDER INGRAM*, PHILOMATH.

EDINBURGH:
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DR SIMSON'S

PREFACE.

THE opinions of the Moderns, concerning the Author of the Elements of Geometry, which go under Euclid's name, are very different, and contrary to one another. Peter Ramus aſcribes the Propoſitions, as well as their Demonſtrations, to Theon; others think the Propoſitions to be Euclid's, but that the Demonſtrations are Theon's; and others maintain, that all the Propoſitions and their Demonſtrations are Euclid's own. John Buteo and Sir Henry Savile are the authors of greateſt note who aſſert this laſt, and the greater part of Geometers have ever ſince been of this opinion, as they thought it the moſt probable. Sir Henry Savile, after the ſeveral arguments he brings to prove it, makes this concluſion, (p. 13. Prælect.) "That, excepting a "very few Interpolations, Explications, and Ad-"ditions, Theon altered nothing in Euclid."

But, by often conſidering and comparing together the Definitions and Demonſtrations, as they are in the Greek editions we now have, I found that Theon, or whoever was the Editor of the preſent Greek text, by adding ſome things, ſuppreſſing others, and mixing his own with Euclid's Demonſtrations, had changed more things to the worſe than is commonly ſuppoſed, and thoſe not of ſmall moment, eſpecially in the Fifth and Eleventh Books of the Elements, which this Editor has greatly vitiated; for inſtance, by ſubſtituting a ſhorter, but in ſufficient Demonſtration of the 18th Propoſition of the 5th Book, in place of the legitimate one which Euclid had given; and by taking out of this Book, beſides other things, the good Definition which Eudoxus or Euclid had given of Compound Ratio, and giving an abſurd one in place of it, in the 5th Definition of the 6th Book, which neither Euclid, Archimedes, Appollonius, nor any Geometer before Theon's time, ever made uſe of, and of which there is not to be found the leaſt appearance in any of their writings; and, as this Definition did much embarraſs beginners, and is quite uſeleſs, it is now thrown out of the Elements, and another, which, without doubt, Euclid had given, is put in its proper place among the Definitions of the 5th Book, by which the doctrine of Compound Ratios is rendered plain and eaſy. Beſides, among the Definitions of the 11th Book, there is this, which is the 10th, viz. "Equal and ſimilar ſolid "Figures are thoſe which are contained by ſimilar "Planes of the ſame number and magnitude." Now, this Propoſition is a Theorem, not a Definition; becauſe the equality of figures of any kind muſt be demonſtrated, and not aſſumed; and therefore, though this were a true Propoſition, it ought to have been demonſtrated. But, indeed, this Propoſition, which makes the 10th Definition of the 11th Book, is not true univerſally, except in the caſe in which each of the ſolid angles of the figures is contained by no more than three plane angles; for, in other caſes, two ſolid figures may be contained by ſimilar planes of the ſame number and magnitude, and yet be unequal to one another; as ſhall be made evident in the Notes ſubjoined to theſe Elements. In like manner, in the Demonſtration of the 26th Propoſition of the 11th Book, it is taken for granted, that thoſe ſolid angles are equal to one another which are contained by plane angles of the ſame number and magnitude, placed in the ſame order; but neither is this univerſally true, except in the caſe in which the ſolid angles are contained by no more than three plane angles; nor of this caſe is there any Demonſtration in the Elements we now have, though it be quite neceſſary there ſhould be one. Now, upon the 10th Definition of this Book depend the 25th and 28th Propoſitions of it; and upon the 25th and 26th depend other eight, viz. the 27th, 31ſt, 32d, 33d, 34th, 36th, 37th, and 40th of the ſame Book; and the 12th of the 12th Book depends upon the 8th of the ſame; and this 8th, and the Corollary of Propoſition 17th and Propoſition 18th of the 12th Book, depend upon the 9th Definition of the 11th Book, which is not a right Definition; becauſe there may be ſolids contained by the ſame number of ſimilar plane figures, which are not ſimilar to one another, in the true ſenſe of ſimilarity received by all Geometers; and all theſe Propoſitions have, for theſe reaſons, been inſufficiently demonſtrated ſince Theon's time hitherto. Beſides, there are ſeveral other things which have nothing of Euclid's accuracy, and which plainly ſhew, that his Elements have been much corrupted by unſkilful Geometers; and, though theſe are not ſo groſs as the others now mentioned, they ought by no means to remain uncorrected.

Upon theſe accounts, it appeared neceſſary, and I hope will prove acceptable to all lovers of accurate reaſoning, and of Mathematical learning, to remove ſuch blemiſhes, and reſtore the principal Books of the Elements to their original accuracy, as far as I was able; eſpecially ſince theſe Elements are the foundation of a Science by which the inveſtigation and diſcovery of uſeful truths, at leaſt in Mathematical learning, is promoted as far as the limited powers of the mind allow; and which likewiſe is of the greateſt uſe in the arts both of peace and war, to many of which Geometry is abſolutely neceſſary. This I have endeavoured to do, by taking away the inaccurate and falſe reaſonings which unſkilful Editors have put into the place of ſome of the genuine Demonſtrations of Euclid, who has ever been juſtly celebrated as the moſt accurate of Geometers, and by reſtoring to him thoſe things which Theon or others have ſuppreſſed, and which have theſe many ages been buried in oblivion.

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THE PREFACE.

IN the preceding Preface, Dr Simſon has ſhown how much the Elements of Euclid have ſuffered from the Greek Editors; and in the Work, he has corrected many errors, and reſtored ſeveral of Euclid's Demonſtrations; by which means, the Elements are in a great meaſure reſtored to their original accuracy. But there are ſome things of great importance overlooked by him, which need correction; and others, though corrected, are not reſtored to their original accuracy, becauſe his corrections are leſs extenſive than the blemiſhes, or are not adapted to Euclid's deſign. For inſtance, he did not obſerve, that the Demonſtration of the 28th Propoſition of the Eleventh Book was inſufficient, though that Propoſition be the foundation of the principal part of ſolid Geometry; and in correcting the 26th of the ſame Book, he overlooked the deſign of the Propoſition, and inſtead of changing the Enunciation, as he ought to have done, he attempted to accommodate the Demonſtration to the Enunciation, as it is in the Greek, in which he did not ſucceed: likewiſe, in correcting the Definition of ſimilar ſolids, he has gone too far from the text, and changed the order of the Definitions,

tions, which he would have had no occaſion to do, if he had properly attended to Euclid. Again, he very properly changed the Demonſtration of the 13th Propoſition of the Third Book, but he did not take notice, that the alteration he complains of there, is only one of a ſeries of alterations made in every Propoſition from the 9th to it; and in the ſame manner, when he corrected the 5th Propoſition of the Fourth Book, he did not obſerve, that the want which he blames in it, is common to it, with many other Propoſitions throughout the Elements. In the Notes, it ſhall be made evident, that ſome corrections are neceſſary in all theſe inſtances. They are accordingly corrected here, together with ſeveral other errors.

To attempt ſuch alterations as theſe, does not ſeem to need an apology; their neceſſity and uſefulneſs are ſufficiently obvious; and in making them, the author walks in a beaten path. But there is another claſs of alterations introduced, that is, the explanation of obſcurities, which, though not leſs uſeful, are not thought to be ſo neceſſary as the former. As to theſe, it ſeems to be enough, if the expreſſion be more perſpicuous than before, and no other objection lie againſt it than what lay againſt the former; and this, it is hoped, is the caſe at preſent. Thus, the Enunciations of the 7th Propoſition of the Firſt Book, and of the 27th, 28th, and 29th of the Sixth Book, are changed; as are alſo the ſecond Definition of the Sixth Book, and the 5th and 7th Definitions of the Fifth Book, beſides ſeveral others of leſs importance. Now, in all theſe places mentioned, the literal tranſlation from the Greek is acknowledged to be very obſcure, ſo that an alteration can ſcarcely be objected to: and the meaning of the preſent mode of expreſſion is very nearly the ſame with that of the former, and the intention and uſe are exactly the ſame. In the Fifth Book, however, the change of expreſſion made in the Definitions, cauſes a ſimilar change in their application, on which account, in the Demonſtrations, there is ſometimes a different ſtep neceſſary in order to connect them with the Definitions, and ſometimes a difference in the conſtruction, but it is generally made more ſimple than before. Beſides, in this Book, the form of the conſtructions is altered, the multiples being now exhibited, by increaſing the magnitudes, inſtead of being made different magnitudes, as they were before; and thoſe of them that are equimultiples, are marked with the ſame letters: By which means, their dependence upon their magnitudes will be more evident, and the Student will find no difficulty, either in diſcovering the multiples of magnitudes, or in knowing which of them are equimultiples;—things which created conſiderable trouble before. In other reſpects, this Book is the ſame as before, except that the 1ſt, 2d, and 6th Propoſitions are more general, and that the Demonſtrations near the beginning are very fully expreſſed.

It will be ſhown in the Notes, that the Definition of proportionals now given, is almoſt the ſame with the ancient definition; and it is obvious, that it agrees with the modern definition, and is a much better expreſſion of it, than that which is commonly given: and it is as eaſily applied as either of them to the purpoſe of demonſtrating the properties of proportionals; ſo that there does not appear to be any valid objection againſt it.

It was at firſt intended to have given the 12th Book of Euclid entire, and to have annexed ſome uſeful Propoſitions to it: but this deſign is dropt at preſent, becauſe that Book is very prolix, and ſeldom read by beginners; and the additional Propoſitions could not be eaſily deduced from it; and to demonſtrate them, independent of it, would have ſwelled the Book too much. It is therefore thought to be more convenient, eſpecially for beginners, for whoſe uſe this Book is chiefly intended, to demonſtrate the relations of the parallelopiped and priſm to the ſolids, which are the ſubject of this Book, and from them to deduce the principal Propoſitions of the 12th Book, which eaſily flow from them; thus forming a plain and ſhort abridgement of it. In the conſtructions of this Book, the figures inſcribed in the circles are compoſed of rectangles made in the manner of the moderns, but the Demonſtrations are conducted in the manner of the ancients: by which means it is manifeſt, that the principal difference between the ancient and modern methods of exhauſtions, does not lie in the methods themſelves, but in the inaccurate mode of expreſſion uſed by many of the moderns.

The 2d of the 12th Book of Euclid is the 3d of this; and the 5th, 6th, and 7th, are contained in the 5th, and its corollaries; and the 10th, 11th, and 12th, in the 6th and 7th, and their corollaries; and the 18th is the 10th of this; all the other Propoſitions of this Book of Euclid are only ſubſidiary ones.

Beſides theſe alterations, there are ſeveral particular errors corrected in this Work, and many of the Demonſtrations which were formerly given in different caſes, are now made more general, and many others are ſhortened. Likewiſe, a number of uſeful Definitions and Propoſitions are added to the Elements, and ſome uſeleſs ones thrown out. But for a more particular account of theſe things, the reader is referred to the Notes, in which he will alſo find the nature of Geometrical accuracy treated pretty fully; and in the Notes on the 29th Propoſition of the Firſt Book, the 12th Axiom is demonſtrated without any aſſumption, and the other Axioms, which the moderns have attempted to ſubſtitute for it, are particularly conſidered: and in the Notes on the Fifth Book, the Definition of proportionals is deduced from the manner of obtaining our firſt ideas of proportion.

In the Elements of Plane and Spherical Trigonometry, annexed to ſome of the Editions of Dr Simſon's Euclid, ſeveral things were aſſumed without proof, which gave conſiderable trouble to beginners. Theſe Elements are now made more accurate and complete; and a new Lemma is prefixed, which is the foundation of the application of Arithmetic to Geometry. Likewiſe, the nature and uſe of the Trigonometrical Tables are explained after the 2d Propoſition, and their conſtruction is given at the end of Plane Trigonometry, to which alſo there is added a method of finding the ratio of the circumference of a circle to its diameter; for without the knowledge of theſe things, Trigonometry cannot be fully underſtood. And in Spherical Trigonometry, the Proportions for reſolving the caſes, which were formerly ſo numerous as to be a burden to the memory, are now reduced to a few general ones, that are as eaſily underſtood and demonſtrated, as any of the particular ones; and eaſy rules are given for preventing the ambiguity of the Solutions. There are alſo many new Demonſtrations given in Trigonometry, much more ſimple than the former ones.

It is acknowledged, that in this performance the brevity affected by ſome modern writers is induſtriouſly avoided, becauſe it is well known, from experience, that inſtead of furthering, it greatly retards the progreſs of Students; and for the ſame reaſon, Algebraic ſymbols are avoided. The Editor is of the ſame opinion with Dr Keil, that "the Elements of all Sciences ought to be handled in the moſt ſimple manner, and not to be involved in Symbols, Notes, or obſcure Principles."

But though words be not uſed ſo ſparingly here as by ſome others, there are very few that could be wanted, without producing obſcurity: and as to tedioufneſs, ſo often complained of, it is rather in appearance than reality; for the arrangement is ſo happily contrived, as almoſt always to admit the ſimpleſt conſtructions and demonſtrations that can be given: and in all the firſt Six Books, there are not above half a dozen of Propoſitions that could be omitted, without a loſs to Geometry. Nor is the method of handling ſolids in the 11th Book ſo tedious as is alledged: for if we omit the 22d, 23d, 26th, and 27th, together with all thoſe after the 34th, as is uſually done by the moderns, the reſt are ſtill accurately demonſtrated without them; and thus, the number of the Propoſitions concerning ſolids is reduced to ten, with two Corollaries, a number as ſmall as has ever been uſed, or can reaſonably be expected in treating ſuch a copious ſubject.

At the end, there is added a Treatiſe of Practical Geometry, a ſubject to which the attention of Students is almoſt always directed, immediately after they have read the Elements.

By Practical Geometry is here meant, the method of expreſſing the magnitude of lines, ſuperficies, &c. by means of the meaſures in common uſe, ſuch as inches, feet, &c. for which purpoſe, it is aſſumed, that the magnitudes concerned are all commenſurable, or rather, that they exceed commenſurable magnitudes, by differences too inconſiderable to be taken notice of; ſo that, when a meaſure is applied, for example, to a line, that line is ſuppoſed to contain the meaſure, or ſome part of it, a certain number of times, without any remainder deſerving notice. In this Treatiſe, the Demonſtrations are as accurate as in the Elements, but they are not quite ſo full, becauſe the Student is now better acquainted with the nature of Demonſtration, and its peculiar mode of expreſſion, than when he was reading the Elements. It is very ſhort, being only an introduction to Menſuration, Surveying, Guaging, &c. and is not intended to ſuperſede the peruſal of more complete Treatiſes on theſe ſubjects.

ERRATA.

Page 18. line 24. from the foot, omitted (margin) a 5. 1,
—— 93. — 3. from the top, *for* AC is *read* AG is
—— 136. — 12. from the foot, *for* tirangle *read* triangle
—— 196. — 15. from the foot, *for* ; *read* :
—— 196. — 13. from the foot, *for* HBC. And *read* HBC, and
—— 233. — 31. from the top, *for* propoſitions, *read* propoſition,
—— 235. — 14. from the foot, *for* ſides *read* ſide
—— 237. — 2. from the top, *for* any angle *read* an angle
—— 237. — 27. from the top, *for* DGE *read* DGF
—— 241. — 25. from the foot, *for* of P. 4. *read* by P. 4.
—— 242. — 22. from the foot, *for* a circle *read* the circle
—— 247. — 12. from the top, *for* ET. (margin) *read* P.T.
—— 266. — 11. from the top, *for* AE *read* AF
—— 277. — 16.&43. from the top, *for* Proctus *read* Proclus
—— 299. — 17. from the top, *for* altogether. *read* all together.
—— 326. — 4. from the foot, *for* 32d *read* 2. Cor. 15.

Beſides theſe, in the 4th line of the Note to PROB. IX. in p. 320. there is ADE, inſtead of ADB, and Pl. I. Fig. 16. is omitted in the margin oppoſite to PROB. X.; and in p. 325. in the firſt line of the laſt column of the Table, there is 13, inſtead of 18.

THE
ELEMENTS
OF
EUCLID.

BOOK I.

DEFINITIONS.

I.

A POINT has ſituation, but not magnitude. See Notes.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A ſtraight line is that which lies evenly between its extreme points.

V.

A ſuperficies is that which hath only length and breadth.

VI.

The extremities of a ſuperficies are lines.

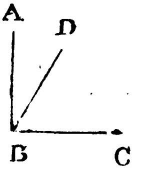
VII.

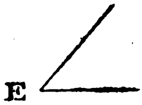
A plane ſuperficies is that in which any two points being taken, the ſtraight line between them lies wholly in that ſuperficies.

VIII. Omitted.

IX.

A plane rectilineal angle is the inclination of two ſtraight lines to one another, which meet together, but are not in the ſame ſtraight line.

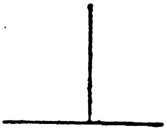




‘ *N. B.* When ſeveral angles are at one point B, any one
‘ of them is expreſſed by three letters, of which the letter that
‘ is at the vertex of the angle, that is, at the point in which
‘ the ſtraight lines that contain the angle meet one another, is
‘ put between the other two letters, and one of theſe two is
‘ ſomewhere upon one of thoſe ſtraight lines, and the other
‘ upon the other line: Thus, the angle which is contained by
‘ the ſtraight lines AB, CB is named the angle ABC, or CBA;
‘ that which is contained by AB, DB is named the angle ABD,
‘ or DBA; and that which is contained by DB, CB is called
‘ the angle DBC, or CBD; but, if there be only one angle at
‘ a point, it may be expreſſed by a letter placed at that point;
‘ as the angle at E.’

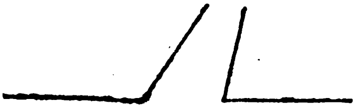
X.

When a ſtraight line ſtanding on another ſtraight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the ſtraight line which ſtands on the other is called a perpendicular to it.



XI.

An obtuſe angle is that which is greater than a right angle.



XII.

An acute angle is that which is leſs than a right angle.

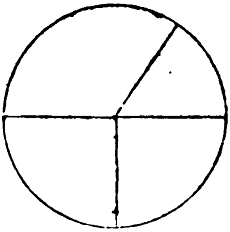
XIII. Omitted.

XIV.

A figure is that which is incloſed by one or more boundaries.

XV.

A circle is a plane figure contained by one line, which is called the circumference, and is ſuch that all ſtraight lines drawn from a certain point within the figure to the circumference, are equal to one another:



XVI.

And this point is called the centre of the circle.

Definition A.

Any ſtraight line drawn from the center to the circumference of a circle, is called a Radius.

XVII.

A diameter of a circle is a ſtraight line drawn through the centre, and terminated both ways by the circumference.

XVIII.

A ſemicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX. Omitted.

XX.

Rectilineal figures are thoſe which are contained by ſtraight lines.

XXI.

Trilateral figures, or triangles, by three ſtraight lines.

XXII.

Quadrilateral, by four ſtraight lines.

XXIII.

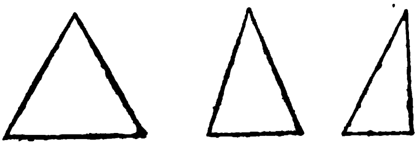
Multilateral figures, or polygons, by more than four ſtraight lines.

XXIV.

Of three-ſided figures, an equilateral triangle is that which has three equal ſides.

XXV.

An Iſoſceles triangle, is that which has only two ſides equal.



XXVI.

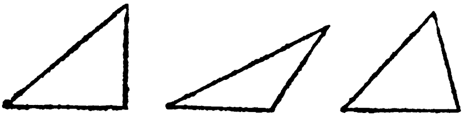
A ſcalene triangle, is that which has three unequal ſides.

XXVII.

A right angled triangle, is that which has a right angle.

XXVIII.

An obtuſe angled triangle, is that which has an obtuſe angle.

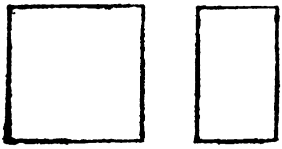


XXIX.

An acute angled triangle, is that which has three acute angles.

XXX.

Of four-ſided figures, a ſquare is that which has all its ſides equal, and all its angles right angles.

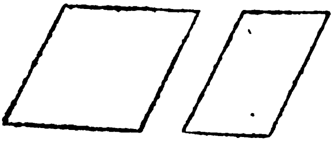


XXXI.

An oblong, is that which has all its angles right angles, but has not all its ſides equal.

XXXII.

A rhombus, is that which has all its ſides equal, but its angles are not right angles.



XXXIII.

A rhomboid, is that which has its oppoſite ſides equal to one another, but all its ſides are not equal, nor its angles right angles.

XXXIV.

All other four-ſided figures, beſides theſe, are called Trapeziums.

XXXV.

Parallel ſtraight lines, are ſuch as are in the ſame plane, and which, being produced ever ſo far both ways, do not meet.

POSTULATES.

I.

LET it be granted that a ſtraight line may be drawn from any one point to any other point.

II.

That a terminated ſtraight line may be produced to any length in a ſtraight line.

III.

And that a circle may be deſcribed from any centre, at any diſtance from that centre.

AXIOMS.

I.

THINGS which are equal to the ſame, are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the ſame, are equal to one another.

VII.

Things which are halves of the ſame, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the ſame ſpace, are equal to one another.

IX.

The whole is greater than its part.

X.

Straight lines which do not coincide, cannot meet one another in more than one point. See N.

XI.

All right angles are equal to one another.

XII.

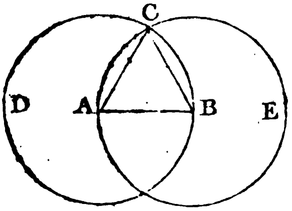
" If a ſtraight line meets two ſtraight lines, ſo as to make the " two interior angles on the ſame ſide of it taken together, " leſs than two right angles, theſe ſtraight lines, being con" tinually produced, ſhall at length meet upon that ſide on " which are the angles which are leſs than two right angles. " See the notes on Prop. 29. of Book I."

PROPO-

PROPOSITION I. PROBLEM.

TO deſcribe an equilateral triangle upon a given finite ſtraight line.

Let AB be the given ſtraight line; it is required to deſcribe an equilateral triangle upon it.



From the centre A, at the diſtance AB, deſcribe [a] the circle BCD; and from the centre B, at the diſtance BA, deſcribe the circle ACE; and from the point C, in which the circles cut one another, draw the ſtraight lines [b] CA, CB to the points A, B; ABC ſhall be an equilateral triangle.

a 3. Poſtulate.

b 1. Poſt.

Becauſe the point A is the centre of the circle BCD, AC is equal [c] to AB; and becauſe the point B is the centre of the circle ACE, BC is equal to BA: But it has been proved, that CA is equal to AB; therefore CA, CB are each of them equal to AB; but things which are equal to the ſame, are equal to one another [d]; therefore CA is equal to CB; wherefore CA, AB, BC, are equal to one another; and the triangle ABC is therefore equilateral, and it is deſcribed upon the given ſtraight line AB. Which was required to be done.

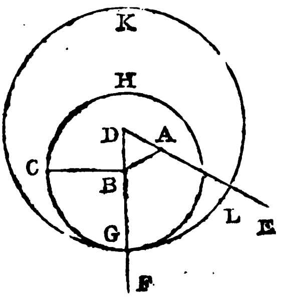
c 15th Definition.

d 1ſt Axiom.

PROP. II. PROB.

FROM a given point to draw a ſtraight line equal to a given ſtraight line.

Let A be the given point, and BC the given ſtraight line; it is required to draw from the point A a ſtraight line equal to BC.



From the point A to B draw [a] the ſtraight line AB; and upon it deſcribe [b] the equilateral triangle DAB, and produce [c] the ſtraight lines DA, DB, to E and F; from the centre B, at the diſtance BC, deſcribe [d] the circle CGH; and from the centre D, at the diſtance DG, deſcribe the circle GKL. AL ſhall be equal to BC.

a 1. Poſt.

b 1. 1.

c 2. Poſt.

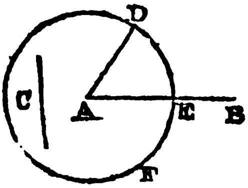
d 3. Poſt.

Becauſe the point B is the centre of the circle CGH, BC is equal [e] to BG; and becauſe D is the centre of the circle GKL, DL is equal to DG, and DA, DB, parts of them, are equal; therefore the remainder AL is equal to the remainder [f] BG: But it has been ſhewn, that BC is equal to BG; wherefore AL and BC are each of them equal to BG; and things that are equal to the ſame, are equal to one another; therefore the ſtraight line AL is equal to BC. Wherefore from the given point A a ſtraight line AL has been drawn equal to the given ſtraight line BC. Which was to be done.

[e] 15. Def.
[f] 3. Ax.

PROP. III. PROB.

FROM the greater of two given ſtraight lines to cut off a part equal to the leſs.



Let AB and C be the two given ſtraight lines, whereof AB is the greater: It is required to cut off from AB, the greater, a part equal to C, the leſs.

From the point A draw [a] the ſtraight line AD equal to C; and from the centre A, at the diſtance AD, deſcribe [b] the circle DEF: And becauſe A is the centre of the circle DEF, AE is equal to AD; but the ſtraight line C is likewiſe equal to AD; whence AE and C are each of them equal to AD; wherefore the ſtraight line AE is equal to [c] C, and from AB, the greater of two ſtraight lines, a part AE has been cut off equal to C the leſs. Which was to be done.

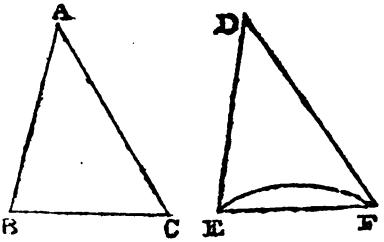
[a] 2. 1.
[b] 3. Poſt.
[c] 1. Ax.

PROP. IV. THEOREM.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each; and have likewiſe the angles contained by thoſe ſides equal to one another; they ſhall likewiſe have their baſes, or *third ſides*, equal; and the two triangles ſhall be equal; and their other angles ſhall be equal, each to each, viz. thoſe to which the equal ſides are oppoſite.

Let ABC, DEF be two triangles, which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz.

Book I. AB to DE, and AC to DF; and the angle BAC equal to the angle EDF; the baſe BC ſhall be equal to the baſe EF; and the triangle ABC to the triangle DEF; and the other angles, to which the equal ſides are oppoſite, ſhall be equal each to each, viz. the angle ABC to the angle DEF, and the angle ACB to DFE.



For, if the triangle ABC be applied to DEF, ſo that the point A may be on D, and the ſtraight line AB upon DE, the point B ſhall coincide with the point E, becauſe AB is equal to DE; and AB coinciding with DE, AC ſhall coincide with DF, becauſe the angle BAC is equal to the angle EDF; wherefore alſo the point C ſhall coincide with the point F, becauſe the ſtraight line AC is equal to DF: But the point B coincides with the point E. Therefore the baſe BC ſhall coincide with the
a 10. Ax. baſe EF a, and be equal to it b. Wherefore the whole triangle ABC ſhall coincide with the whole triangle DEF, and be equal
b 8. Ax. to it b; and the other angles of the one ſhall coincide with the remaining angles of the other, and be equal to them, viz. the angle ABC to the angle DEF, and the angle ACB to DFE. Therefore, if two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe the angles contained by thoſe ſides equal to one another, their baſes ſhall likewiſe be equal, and the triangles be equal, and their other angles to which the equal ſides are oppoſite, ſhall be equal, each to each. Which was to be demonſtrated.

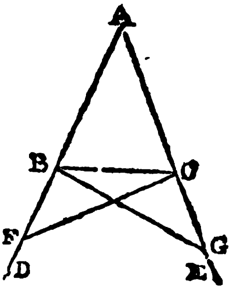
PROP. V. THEOR.

THE angles at the baſe of an Iſoſceles triangle are equal to one another; and, if the equal ſides be produced, the angles upon the other ſide of the baſe ſhall be equal.

Let ABC be an Iſoſceles triangle, of which the ſide AB is equal to AC, and let the ſtraight lines AB, AC be produced to D and E, the angle ABC ſhall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD, take any point F, and from AE, the greater, cut off
a 3. 1. AG equal a to AF, the leſs, and join FC, GB.

Becauſe AF is equal to AG, and AB to AC, the two ſides FA, AC are equal to the two GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB; therefore the baſe FC is equal [b] to the baſe GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal [b] to the remaining angles of the other, each to each, to which the equal ſides are oppoſite, viz. the angle ACF to the angle ABG, and the angle AFC to the angle AGB: And becauſe the whole AF is equal to the whole AG, of which the parts AB, AC are equal; the remainder BF ſhall be equal [c] to the remainder CG; and FC was proved to be equal to GB; therefore the two ſides BF, FC are equal to the two CG, GB, each to each; and the angle BFC is equal to the angle CGB, and the baſe BC is common to the two triangles BFC, CGB; wherefore the triangles are equal [b], and their remaining angles, each to each, to which the equal ſides are oppoſite; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG: And ſince it has been demonſtrated, that the whole angle ABG is equal to the whole ACF, the parts of which, the angles CBG, BCF are alſo equal; the remaining angle ABC is therefore equal to the remaining angle ACB, which are the angles at the baſe of the triangle ABC: And it has alſo been proved, that the angle FBC is equal to the angle GCB, which are the angles upon the other ſide of the baſe. Therefore the angles at the baſe, &c. Q. E. D.



b 4. 1.

c 3. Ax.

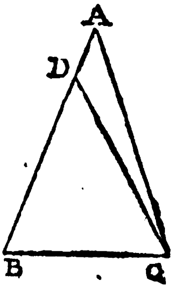
Corollary. Hence every equilateral triangle is alſo equiangular.

PROP. VI. THEOR.

If two angles of a triangle be equal to one another, the ſides alſo which ſubtend, *or are oppoſite to*, the equal angles, ſhall be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB; the ſide AB is equal to the ſide AC.

For if AB be not equal to AC, one of them is greater than the other: Let AB be the greater, and from it cut [a] off DB equal to AC, the leſs, and join DC; therefore, becauſe in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two ſides DB, BC are equal to the two AC, CB, each to each; and the angle DBC is equal to the angle ACB; therefore the baſe DC is equal to the baſe AB, and the triangle DBC is equal



a 3. 1.

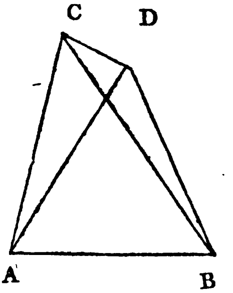
to the triangle [b] ACB, the leſs to the greater; which is abſurd. Therefore AB is not unequal to AC, that is, it is equal to it.
b. 4. 1. Wherefore, if two angles, &c. Q. E. D.

Cor. Hence every equiangular triangle is alſo equilateral.

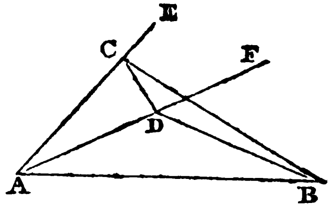
PROP. VII. THEOR.

See N. If there be two triangles upon the ſame baſe, and on the ſame ſide of it, that have their ſides, which are terminated in one extremity of the baſe equal to one another, they ſhall not have their ſides equal, which are terminated in the other extremity.

Let there be two triangles ACB, ADB, upon the ſame baſe AB, and upon the ſame ſide of it, which have their ſides CA, DA, terminated in the extremity A of the baſe, equal to one another; their ſides CB, DB, that are terminated in B, ſhall not be equal.



Join CD; then in the caſe in which the vertex of each of the triangles is without the other triangle, becauſe AC is equal to AD, the angle ACD is
1 equal [a] to the angle ADC: But the angle ACD is greater than the angle BCD; therefore the angle ADC is greater alſo than BCD; much more then is the angle BDC greater than the angle BCD. But if CB were equal to DB, the angle BDC, would be equal [a] to the angle BCD; and it has been demonſtrated to be greater than it; which is impoſſible; therefore CB cannot be equal to DB.



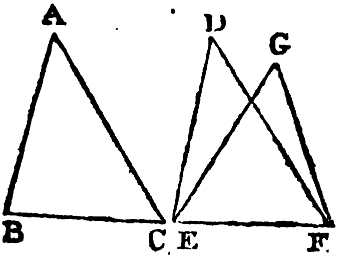
But, if one of the vertices, as D, be within the other triangle ACB; produce AC, AD to E, F; therefore, becauſe AC is equal to AD, in the triangle ACD, the angles ECD, FDC upon the other ſide of the baſe CD, are equal [a] to one another; but the angle ECD is greater than the angle BCD; wherefore the angle FDC is likewiſe greater than BCD; much more then is the angle BDC greater than the angle BCD. But if CB were equal to DB, the angle BDC would be equal [a] to the angle BCD; and BDC has been proved to be greater than the ſame

BCD; which is impoſſible. Therefore CB cannot be equal to DB. The caſe in which the vertex of one triangle is upon a ſide of the other, needs no demonſtration. Wherefore, if there be two triangles, &c. Q. E. D.

PROP. VIII. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe their baſes equal; the angle which is contained by the two ſides of the one, ſhall be equal to the angle contained by the two ſides equal to them of the other.

Let ABC, DEF be two triangles, having the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB to DE, and AC to DF; and alſo the baſe BC equal to the baſe EF. The angle BAC is equal to the angle EDF.



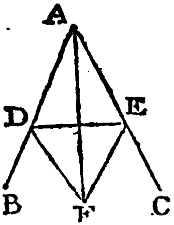
For, if the triangle ABC be applied to DEF, ſo that the point B be on E, and the ſtraight line BC upon EF; the point C ſhall alſo coincide with the point F, becauſe BC is equal to EF; therefore BC coinciding with EF, BA, and AC ſhall coincide with ED and DF; for, if the baſe BC coincides with the baſe EF, but the ſides BA, CA do not coincide with the ſides ED, FD, but have a different ſituation, as EG, FG; then, upon the ſame baſe EF, and upon the ſame ſide of it, there can be two triangles that have their ſides, which are terminated in one extremity of the baſe, equal to one another, and likewiſe their ſides terminated in the other extremity; but this is impoſſible[a]: a 7. 1.
Therefore, if the baſe BC coincides with the baſe EF, the ſides BA, AC cannot but coincide with the ſides ED, DF; wherefore, likewiſe the angle BAC coincides with the angle EDF, and is equal[b] to it. Therefore, if two triangles, &c. b 8. Ax
Q. E. D.

PROP. IX. PROB.

TO biſect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle, it is required to biſect it.

a 3. 1. b 1. 1. Take any point D in AB, and from AC cut [a] off AE equal to AD; join DE, and upon it deſcribe [b] an equilateral triangle DEF; then join AF; the ſtraight line AF biſects the angle BAC.



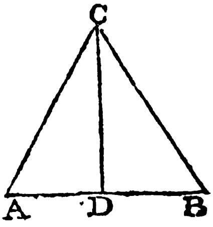
c 8. 1. Becauſe AD is equal to AE, and AF is common to the two triangles DAF, EAF; the two ſides DA, AF are equal to the two ſides EA, AF, each to each; and the baſe DF is alſo equal to the baſe EF; therefore the angle DAF is equal [c] to the angle EAF; wherefore the given rectilineal angle BAC is biſected by the ſtraight line AF. Which was to be done.

PROP. X. PROB.

TO biſect a given finite ſtraight line, that is, to divide it into two equal parts.

Let AB be the given ſtraight line; it is required to divide it into two equal parts.

a 1. 1. b 9. 1. Deſcribe [a] upon it an equilateral triangle ABC, and biſect [b] the angle ACB by the ſtraight line CD. AB is cut into two equal parts in the point D.



c 4. 1. Becauſe AC is equal to CB, and CD common to the two triangles ACD, BCD; the two ſides AC, CD are equal to BC, CD, each to each; and the angle ACD is alſo equal to the angle BCD; therefore the baſe AD is equal to the baſe [c] DB, and the ſtraight line AB is divided into two equal parts in the point D. Which was to be done.

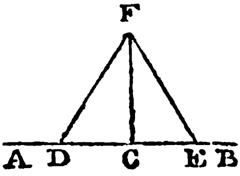
PROP. XI. PROB.

TO draw a ſtraight line at right angles, to a given ſtraight line, from a given point in the ſame.

Let AB be a given ſtraight line, and C a point given in it; it is required to draw a ſtraight line from the point C, at right angles to AB.

a 3. 1. b 1. 1. Take any point D in AC, and [a] make CE equal to CD, and upon DE deſcribe [b] the equilateral triangle DFE, and join FC; the ſtraight line FC drawn from the given point C, is at right angles to the given ſtraight line AB.

c 8. 1. Becauſe DC is equal to CE, and FC common to the two triangles DCF, ECF; the two ſides DC, CF are equal to the two EC, CF, each to each; and the baſe DF is alſo equal to the baſe EF; therefore the angle DCF is equal [c] to the angle ECF;



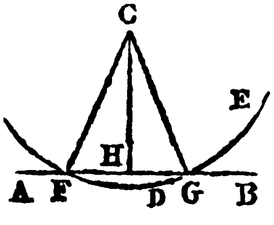
ECF; and they are adjacent angles. But, when the adjacent angles which one ſtraight line makes with another ſtraight line, are equal to one another, each of them is called a right [d] angle; therefore each of the angles DCF, ECF is a right angle. Wherefore, from the given point C, in the given ſtraight line AB, FC has been drawn at right angles to AB. Which was to be done.

d 10. Def.

PROP. XII. PROB.

TO draw a ſtraight line perpendicular to a given ſtraight line of an unlimited length, from a given point without it.

Let AB be the given ſtraight line, which may be produced to any length both ways, and let C be a point without it. It is required to draw a ſtraight line perpendicular to AB from the point C.



Take any point D upon the other ſide of AB, and from the centre C, at the diſtance CD, deſcribe [a] the circle EGF meeting AB in F, G; and biſect [b] FG in H, and join CF, CH, CG; the ſtraight line CH, drawn from the given point C, is perpendicular to the given ſtraight line AB.

a 3. Poſt.
b 10. 1.

Becauſe FH is equal to HG, and HC common to the two triangles FHC, GHC, the two ſides FH, HC are equal to the two GH, HC, each to each; and the baſe CF is alſo equal [c] to the baſe CG; therefore the angle CHF is equal [d] to the angle CHG; and they are adjacent angles; but when a ſtraight line ſtanding on a ſtraight line makes the adjacent angles equal to one another, each of them is a right angle, and the ſtraight line which ſtands upon the other, is called a perpendicular to it; therefore from the given point C, a perpendicular CH has been drawn to the given ſtraight line AB. Which was to be done.

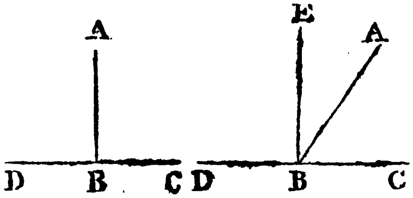
c 15. Def.
d 8. 1.

PROP. XIII. THEOR.

THE angles which one ſtraight line makes with another upon the one ſide of it, are either two right angles, or are together equal to two right angles.

Let the ſtraight line AB make with CD, upon one ſide of it, the angles CBA, ABD; theſe are either two right angles, or are together equal to two right angles.

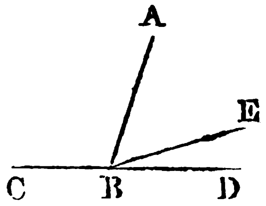
For, if the angle CBA be equal to ABD, each of them is a



a Def. 10. right [a] angle; but, if not, from the point B draw BE at right b 11. 1. angles [b] to CD; therefore the angles CBE, EBD are two right angles [a]; and becauſe CBE is equal to the two angles CBA, ABE together, add the angle EBD to each of theſe equals; therefore c 2. Ax. the angles CBE, EBD are equal [c] to the three angles CBA, ABE, EBD. Again, becauſe the angle DBA, is equal to the two angles DBE, EBA, add to theſe equals the angle ABC; therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC; but the angles CBE, EBD have been demonſtrated to be equal to the ſame three angles; and things d 1. Ax. that are equal to the ſame, are equal [d] to one another; therefore the angles CBE, EBD are equal to the angles DBA, ABC: but CBE, EBD are two right angles; therefore, DBA, ABC are together equal to two right angles. Wherefore, when a ſtraight line, &c. Q. E. D.

PROP. XIV. THEOR.

IF, at a point in a ſtraight line, two other ſtraight lines, upon the oppoſite ſides of it, make the adjacent angles together equal to two right angles, theſe two ſtraight lines ſhall be in one and the ſame ſtraight line.



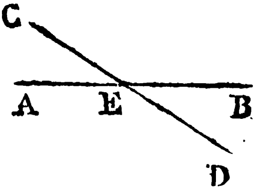
At the point B in the ſtraight line AB, let the two ſtraight lines BC, BD, upon the oppoſite ſides of AB, make the adjacent angles ABC, ABD equal together to two right angles. BD is in the ſame ſtraight line with CB.

For, if BD be not in the ſame ſtraight line with CB, let BE be in the ſame ſtraight line with it; therefore, becauſe the ſtraight line AB makes angles with the ſtraight line CBE, upon one a 13. 1. ſide of it, the angles ABC, ABE are together equal [a] to two right angles; but the angles ABC, ABD are likewiſe together equal to two right angles; therefore the angles CBA, ABE are equal to the angles CBA, ABD: Take away the common angle ABC, and the remaining angle ABE is equal [b] to the remaining angle ABD, the leſs to the greater, which is impoſſible; therefore BE is not in the ſame ſtraight line with BC. And, in like manner, it may be demonſtrated, that no other can be in the ſame ſtraight line with it but BD, which therefore is in the ſame ſtraight line with CB. Wherefore, if at a point, &c. Q. E. D. b 3. Ax.

PROP. XV. THEOR.

IF two ſtraight lines cut one another, the vertical, *or oppoſite*, angles ſhall be equal.

Let the two ſtraight lines AB, CD cut one another in the point E; the angle AEC ſhall be equal to the angle DEB, and CEB to AED.



Becauſe the ſtraight line AE makes with CD the angles CEA, AED, theſe angles are together equal [a] to two right angles. Again, becauſe the ſtraight line DE makes with AB the angles AED, DEB, theſe alſo are together equal [a] to two right angles; and CEA, AED have been demonſtrated to be equal to two right angles; wherefore the angles CEA, AED are equal to the angles AED, DEB. Take away the common angle AED, and the remaining angle CEA is equal [b] to the remaining angle DEB. In the ſame manner, it can be demonſtrated, that the angles CEB, AED are equal. Therefore, if two ſtraight lines, &c. Q. E. D. a 13. 1. b 3. Ax.

Cor. 1. From this it is manifeſt, that, if two ſtraight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

Cor. 2. And conſequently, that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

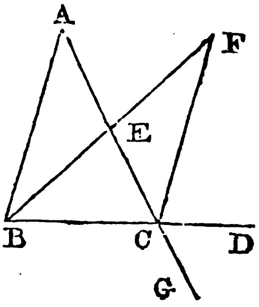
Cor. 3. Likewiſe, if two ſtraight lines CE, ED, on oppoſite ſides of a ſtraight line AB, meet at a point E in it, and make the vertical angles AEC, BED equal to one another, they are in the ſame ſtraight line. For the angles DEB, BEC are equal [c] to AEC, CEB, that is, to two right angles [d]; therefore CE is in the ſame ſtraight line [e] with ED. c 1. Ax. d 13. 1. e 14. 1.

PROP.

PROP. XVI. THEOR.

IF one ſide of a triangle be produced, the exterior angle is greater than either of the interior oppoſite angles.

Let ABC be a triangle, and let its ſide BC be produced to D, the exterior angle ACD is greater than either of the interior oppoſite angles CBA, BAC.

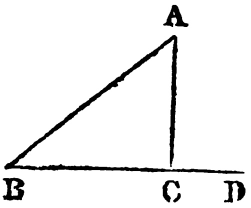


a 10. 1. Biſect[a] AC in E, join BE, and
produce it to F, and make EF equal
to BE; join alſo FC, and produce
AC to G.

Becauſe AE is equal to EC, and
BE to EF; AE, EB are equal to
CE, EF, each to each; and the
b 15. 1. angle AEB is alſo equal[b] to the
angle CEF, becauſe they are vertical
c 4. 1. angles; therefore the baſe AB is[c]
equal to the baſe CF, and the triangle
AEB to the triangle CEF, and the
remaining angles to the remaining angles, each to each, to which
the equal ſides are oppoſite; wherefore the angle BAE is equal
to the angle ECF; but the angle ECD is greater than the angle
ECF; therefore the angle ACD is greater than BAE: In the
ſame manner, if the ſide BC be biſected, it may be demonſtrated,
d 15. 1. that the angle BCG, that is[d], the angle ACD, is greater than
the angle ABC. Therefore, if one ſide, &c. Q. E. D.

PROP. XVII. THEOR.

ANY two angles of a triangle are together leſs than two right angles.



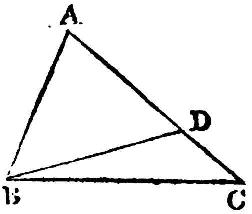
Let ABC be any triangle; any two of its angles together are leſs than two right angles.

Produce BC to D; and becauſe
ACD is the exterior angle of the
a 16. 1. triangle ABC, ACD is greater[a] than
the interior and oppoſite angle ABC;
to each of theſe, add the angle ACB;
therefore the angles ACD, ACB are greater than the angles
b 13. 1. ABC, ACB; but ACD, ACB are together equal[b] to two
right angles; therefore the angles ABC, BCA are leſs than two
right angles. In like manner, it may be demonſtrated, that
BAC,

BAC, ACB, as alſo CAB, ABC are leſs than two right angles. Therefore any two angles, &c. Q. E. D.

PROP. XVIII. THEOR.

THE greater ſide of every triangle is oppoſite to the greater angle.



Let ABC be a triangle, of which the ſide AC is greater than the ſide AB; the angle ABC is alſo greater than the angle BCA.

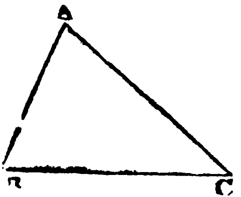
Becauſe AC is greater than AB, make [a] AD equal to AB, and join BD; and becauſe ADB is the exterior angle of the triangle BDC, it is greater [b] than the interior and oppoſite angle DCB; but ADB is equal [c] to ABD, becauſe the ſide AB is equal to the ſide AD; therefore the angle ABD is likewiſe greater than the angle ACB; wherefore much more is the angle ABC greater than ACB. Therefore the greater ſide, &c. Q. E. D.

a 3. 1.
b 16. 1.
c 5. 1.

PROP. XIX. THEOR.

THE greater angle of every triangle is ſubtended by the greater ſide, or *has the greater ſide oppoſite to it.*

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA; the ſide AC is likewiſe greater than the ſide AB.



For, if it be not greater, AC muſt either be equal to AB, or leſs than it; it is not equal, becauſe then the angle ABC would be equal [a] to the angle ACB; but it is not; therefore AC is not equal to AB: neither is it leſs; becauſe then the angle ABC would be leſs [b] than the angle ACB; but it is not; therefore the ſide AC is not leſs than AB: and it has been ſhewn, that it is not equal to AB; therefore AC is greater than AB. Wherefore the greater angle, &c. Q. E. D.

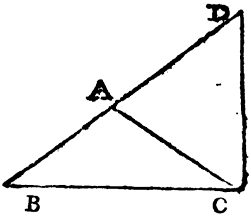
a 5. 1.
b 18. 1.

PROP. XX. THEOR.

ANY two ſides of a triangle are together greater than the third ſide.

Let ABC be a triangle; any two ſides of it together are greater than the third ſide, viz. the ſides BA, AC greater than the ſide BC; and AB, BC greater than AC; and BC, CA greater than AB.

a 3. 1. Produce BA to D, and make [a] AD equal to AC; and join DC.



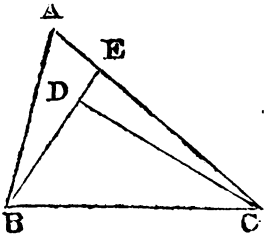
Becauſe DA is equal to AC, the
b 5. 1. angle ADC is likewiſe equal [b] to ACD; but the angle BCD is greater than ACD; therefore BCD is alſo greater than ADC; and becauſe the angle BCD of the triangle DCB is greater than its angle
c 19. 1. BDC, and that the greater [c] ſide is oppoſite to the greater angle; therefore the ſide DB is greater than the ſide BC; but DB is equal to BA and AC; therefore the ſides BA, AC are greater than BC. In the ſame manner, it may be demonſtrated, that the ſides AB, BC are greater than CA, and BC, CA greater than AB. Therefore any two ſides, &c. Q. E. D.

PROP. XXI. THEOR.

IF, from the ends of the ſide of a triangle, there be drawn two ſtraight lines to a point within the triangle, theſe ſhall be leſs than the other two ſides of the triangle, but ſhall contain a greater angle.

Let the two ſtraight lines BD, CD be drawn from B, C, the ends of the ſide BC of the triangle ABC, to the point D within it; BD and DC are leſs than the other two ſides BA, AC of the triangle, but contain an angle BDC greater than the angle BAC.

Produce BD to E; and becauſe two ſides of a triangle are
a 20. 1. greater than the third ſide [a], the two ſides BA, AE of the triangle ABE are greater than BE: To each of theſe, add EC; therefore the ſides BA, AC are greater than BE, EC [b]. Again,
b 2. Ax. becauſe the two ſides CE, ED of the triangle CED are greater than CD, add DB to each of theſe; therefore the ſides CE, EB are greater than CD, DB; but it has been ſhewn, that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC.



Again, becauſe the exterior angle of a triangle is greater than
c 16. 1. the interior and oppoſite angle [c], the exterior angle BDC of the triangle CDE is greater than CED; for the ſame reaſon, the

exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonſtrated, that the angle BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC. Therefore, if from the ends of, &c. Q. E. D.

PROP. XXII. PROB.

TO make a triangle of which the ſides ſhall be equal to three given ſtraight lines, but any two whatever of theſe muſt be greater than the third [a].

See N.

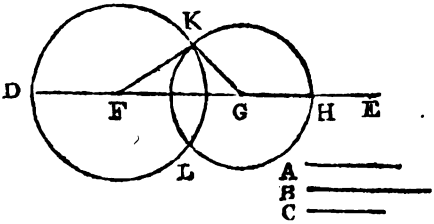
a 20. 1.

Let A, B, C be the three given ſtraight lines, of which any two whatever are greater than the third, viz. A and B greater than C; A and C greater than B; and B and C than A: It is required to make a triangle of which the ſides ſhall be equal to A, B, C, each to each.

Take a ſtraight line DE terminated at the point D, but unlimited towards E, and make [a] DF equal to A, FG to B, and GH equal to C; therefore any two of the ſtraight lines DF, FG, GH are greater than the third: from the centre F, at the diſtance FD, deſcribe [b] the circle DKL; it ſhall cut FH, for FD is leſs than FH: alſo from the centre G, at the diſtance GH, deſcribe another circle HKL; and it ſhall cut GD, for GH is leſs than GD: and the two circles cut one another, becauſe DF, GH are together greater than FG; let them cut one another in K, and join KF, KG; the triangle KFG has its ſides equal to the three ſtraight lines A, B, C.

a 3. 1.

b 3. Poſt.



Becauſe the point F is the centre of the circle DKL, FD is equal [c] to FK; but FD is equal to the ſtraight line A; therefore FK is equal to A: Again, becauſe G is the centre of the circle LKH, GH is equal [c] to GK; but GH is equal to C; therefore alſo GK is equal to C; and FG is equal to B; therefore the three ſtraight lines KF, FG, GK are equal to the three A, B, C: And therefore the triangle KFG has its three ſides KF, FG, GK, equal to the three given ſtraight lines A, B, C. Which was to be done.

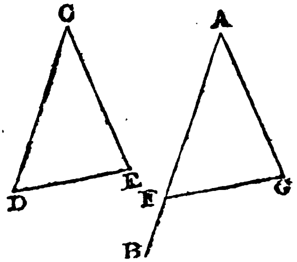
c 15. Def.

PROP. XXIII. PROB.

AT a given point in a given ſtraight line, to make a rectilineal angle equal to a given rectilineal angle.

Let

Book I. Let AB be the given ſtraight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A, in the given ſtraight line AB, that ſhall be equal to the given rectilineal angle DCE.



Take in CD, CE, any points
a 22. 1. D, E, and join DE; and make a the triangle AFG, the ſides of which ſhall be equal to the three ſtraight lines CD, DE, CE, ſo that CD be equal to AF, CE to AG, and DE to FG; and becauſe DC, CE are equal to FA, AG, each to each, and the baſe DE to the baſe FG; the angle DCE
b 8. 1. is equal b to the angle FAG: Therefore, at the given point A, in the given ſtraight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

PROP. XXIV. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the angle contained by the two ſides of one of them, greater than the angle contained by the two ſides equal to them, of the other; the baſe of that which has the greater angle, ſhall be greater than the baſe of the other.

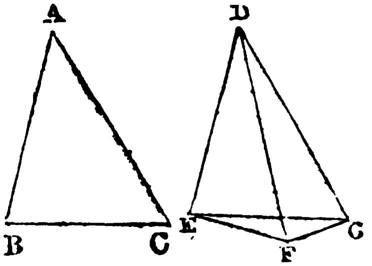
Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; the baſe BC is alſo greater than the baſe EF.

Of the two ſides DE, DF, let DE be the ſide which is not greater than the other, and at the point D, in the ſtraight line
a 23. 1. DE, make a the angle EDG equal to the angle BAC; and
b 3. 1. make DG equal b to AC or DF, and join EG, GF.

Becauſe AB is equal to DE, and AC to DG, the two ſides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is alſo equal to the angle EDG; therefore the baſe
c 4. 1. BC is equal c to the baſe EG; and becauſe DG is equal to DF,
d 5. 1. the angle DFG is equal d to the angle DGF; but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF; and much more is the angle EFG greater than EGF; and becauſe the angle EFG of the triangle EFG is greater than its angle EGF, and that the

c 19. 1.

the greater [e] ſide is oppoſite to the greater angle; the ſide EG is therefore greater than the ſide EF; but EG is equal to BC; and therefore alſo BC is greater than EF. Therefore, if two triangles, &c. Q. E. D.

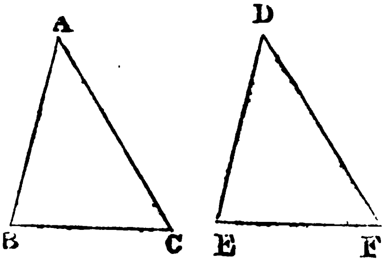


PROP. XXV. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the baſe of the one greater than the baſe of the other; the angle alſo contained by the ſides of that which has the greater baſe, ſhall be greater than the angle contained by the ſides equal to them, of the other.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the baſe CB is greater than the baſe EF; the angle BAC is likewiſe greater than the angle EDF.

For, if it be not greater, it muſt either be equal to it, or leſs; but the angle BAC is not equal to the angle EDF, becauſe then the baſe BC would be equal [a] to EF; but it is not; therefore the angle BAC is not equal to the angle DEF: neither is it leſs; becauſe then the baſe BC would be leſs [b] than the baſe EF; but it is not; therefore the angle BAC is not leſs than the angle EDF: and it was ſhewn that it is not equal to it; therefore the angle BAC is greater than the angle EDF. Wherefore, if two triangles, &c. Q. E. D.



a 4. 1.

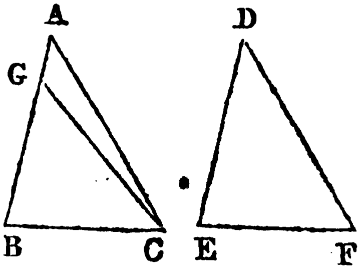
b 24. 1.

PROP.

PROP. XXVI. THEOR.

IF two triangles have two angles of the one equal to two angles of the other, each to each; and one ſide equal to one ſide, viz. either the ſides adjacent to the equal angles, or the ſides oppoſite to equal angles in each: then ſhall the other ſides be equal, each to each; and alſo the third angle of the one to the third angle of the other.

Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, viz. ABC to DEF, and BCA to EFD; alſo one ſide equal to one ſide; and firſt let thoſe ſides be equal which are adjacent to the angles that are equal in the two triangles, viz. BC to EF; the other ſides ſhall be equal, each to each, viz. AB to DE, and AC to DF; and thet hird angle BAC to the third angle EDF.

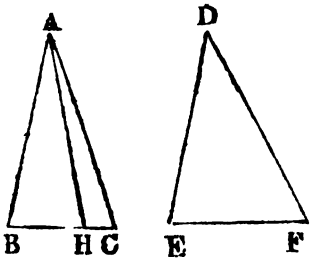


For, if AB be not equal to DE, one of them muſt be the greater. Let AB be the greater of the two, and make BG equal to DE, and join GC; therefore, becauſe BG is equal to DE, and BC to EF, the two ſides GB, BC are equal to the two DE, EF, each to each; and the angle GBC is alſo equal to
a 4. 1. the angle DEF; therefore the baſe GC is equal [a] to the baſe DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal ſides are oppoſite; therefore the angle GCB is equal to the angle DFE; but DFE is, by the hypotheſis, equal to the angle BCA; wherefore alſo the angle BCG is equal to the angle BCA, the leſs to the greater, which is impoſſible; therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF; the baſe therefore AC is equal [a] to the baſe DF, and the third angle BAC to the third angle EDF.

Next, let the ſides which are oppoſite to equal angles in each triangle be equal to one another, viz. AB to DE; likewiſe in this caſe, the other ſides ſhall be equal, AC to DF, and BC to EF; and alſo the third angle BAC to the third EDF.

For, if BC be not equal to EF, let BC be the greater of them, and make BH equal to EF, and join AH; and becauſe BH is

equal



equal to EF, and AB to DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal angles; therefore the baſe AH is equal to the baſe DF, and the triangle ABH to the triangle DEF, and the other angles ſhall be equal, each to each, to which the equal ſides are oppoſite; therefore the angle BHA is equal to the angle EFD; but EFD is equal to the angle BCA; therefore alſo the angle BHA is equal to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and oppoſite angle BCA; which is impoſſible [b]; wherefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE; therefore the two AB, BC are equal to the two DE, EF, each to each; and they contain equal angles; wherefore the baſe AC is equal to the baſe DF, and the third angle BAC to the third angle EDF. Therefore, if two triangles, &c. Q. E. D.

b 16. 1.

PROP. XXVII, THEOR.

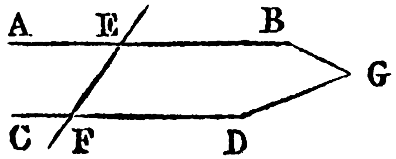
IF a ſtraight line falling upon two other ſtraight lines, makes the alternate angles equal to one another, theſe two ſtraight lines ſhall be parallel.

Let the ſtraight line EF, which falls upon the two ſtraight lines AB, CD, make the alternate angles AEF, EFD equal to one another; AB is parallel to CD.

For, if it be not parallel, AB and CD being produced, ſhall meet either towards B, D, or towards A, C; let them be produced and meet towards B, D in the point G; therefore GEF is a triangle, and its exterior angle AEF is greater [a] than the interior and oppoſite angle EFG; but it is alſo equal to it, which is impoſſible; therefore AB and CD being produced, do not meet towards B, D. In like manner, it may be demonſtrated, that they do not meet towards A, C; but thoſe ſtraight lines which meet neither way, though produced ever ſo far, are parallel [b] to one another. AB therefore is parallel to CD. Wherefore, if a ſtraight line, &c. Q. E. D.

a 16. 1.

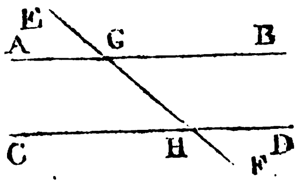
b 35. Def.



PROP.

PROP. XXVIII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the exterior angle equal to the interior and oppoſite upon the ſame ſide of the line; or makes the interior angles upon the ſame ſide together equal to two right angles; the two ſtraight lines ſhall be parallel to one another.



Let the ſtraight line EF, which falls upon the two ſtraight lines AB, CD, make the exterior angle EGB equal to the interior and oppoſite angle GHD upon the ſame ſide; or make the interior angles on the ſame ſide BGH, GHD together equal to two right angles; AB is parallel to CD.

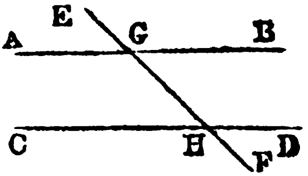
Becauſe the angle EGB is equal to the angle GHD, and the
a 15. 1. angle EGB equal [a] to the angle AGH, the angle AGH is equal
b 27. 1. to the angle GHD; and they are the alternate angles; therefore
c By Hyp. AB is parallel [b] to CD. Again, becauſe the angles BGH, GHD
are equal [c] to two right angles, and that AGH, BGH are alſo
d 13. 1. equal [d] to two right angles; the angles AGH, BGH are equal
to the angles BGH, GHD: Take away the common angle
BGH; therefore the remaining angle AGH is equal to the remaining angle GHD; and they are alternate angles; therefore AB is parallel to CD. Wherefore, if a ſtraight line, &c. Q. E. D.

PROP. XXIX. THEOR.

See the notes on this propoſition.

IF a ſtraight line fall upon two parallel ſtraight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and oppoſite upon the ſame ſide; and likewiſe the two interior angles upon the ſame ſide together equal to two right angles.

Let the ſtraight line EF fall upon the parallel ſtraight lines AB, CD; the alternate angles AGH, GHD are equal to one another; and the exterior angle EGB is equal to the interior and oppoſite, upon the ſame ſide, GHD; and the two interior angles BGH, GHD, upon the ſame ſide, are together equal to two right angles.



For, if AGH be not equal to GHD, one of them muſt be greater than the other; let AGH be the greater; and becauſe the angle AGH is greater than the angle GHD, add to each of them the angle BGH; therefore the angles AGH, BGH are greater than the angles BGH, GHD: but the angles AGH, BGH are equal [a] to two right angles; therefore the angles BGH, GHD are leſs than two right angles: but thoſe ſtraight lines which, with another ſtraight line falling upon them, make the interior angles on the ſame ſide leſs than two right angles, do meet * together, if continually produced; therefore the ſtraight lines AB, CD, if produced far enough, ſhall meet: but they never meet, ſince they are parallel by the hypotheſis; therefore the angle AGH is not unequal to the angle GHD, that is, it is equal to it; but the angle AGH is equal [b] to the angle EGB; therefore likewiſe EGB is equal to GHD; add to each of theſe the angle BGH; therefore the angles EGB, BGH are equal to the angles BGH, GHD: but EGB, BGH are equal [c] to two right angles; therefore alſo BGH, GHD are equal to two right angles. Wherefore, if a ſtraight line, &c. Q. E. D.

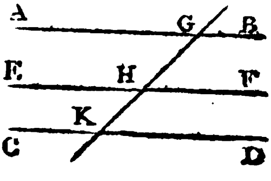
[a] 13. 1.
* 12. Ax. See the notes on this propoſition.
[b] 15. 1.
[c] 13. 1.

PROP. XXX. THEOR.

STRAIGHT lines which are parallel to the ſame ſtraight line are parallel to one another.

Let AB, CD be each of them parallel to EF; AB is alſo parallel to CD.

Let the ſtraight line GHK cut AB, EF, CD; and becauſe GHK cuts the parallel ſtraight lines AB, EF, the angle AGH is equal [a] to the angle GHF. Again, becauſe the ſtraight line GK cuts the parallel ſtraight lines EF, CD, the angle GHF is equal [a] to the angle GKD: and it was ſhewn, that the angle AGK is equal to the angle GHF; therefore alſo AGK is equal to GKD; and they are alternate angles; therefore AB is parallel [b] to CD. Wherefore, ſtraight lines, &c. Q. E. D.

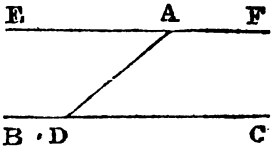


[a] 29. 1.
[b] 27. 1.

PROP. XXXI. PROB.

TO draw a ſtraight line through a given point parallel to a given ſtraight line.

Book I. Let A be the given point, and BC the given ſtraight line; it is required to draw a ſtraight line through the point A, parallel to the ſtraight line BC.



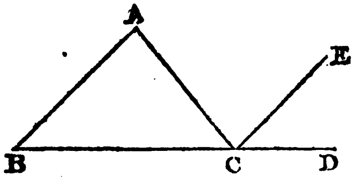
In BC take any point D, and join AD; and at the point A in the a 23. 1. ſtraight line AD make [a] the angle DAE equal to the angle ADC; and produce the ſtraight line EA to F.

Becauſe the ſtraight line AD, which meets the two ſtraight lines BC, EF, makes the alternate angles EAD, ADC equal to b 27. 1. one another, EF is parallel [b] to BC. Therefore the ſtraight line EAF is drawn through the given point A parallel to the given ſtraight line BC. Which was to be done.

PROP. XXXII. THEOR.

IF a ſide of any triangle be produced, the exterior angle is equal to the two interior and oppoſite angles; and the three interior angles of every triangle are equal to two right angles.

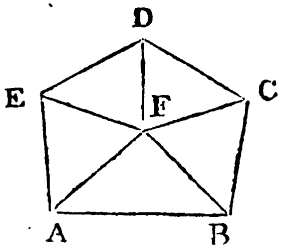
Let ABC be a triangle, and let one of its ſides BC be produced to D; the exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC; and the three interior angles of the triangle, viz. ABC, BCA, CAB are together equal to two right angles.



Through the point C draw a 31. 1. CE parallel [a] to the ſtraight line AB; and becauſe AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are e- b 29. 1. qual [b]. Again, becauſe AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and oppoſite angle ABC; but the angle ACE was ſhown to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC; to theſe equals add the angle ACB, and the angles ACD, ACB are equal to the three angles CBA, c 13. 1. BAC, ACB; but the angles ACD, ACB are equal [c] to two right angles; therefore alſo the angles CBA, BAC, ACB are equal to two right angles. Wherefore, if a ſide of a triangle, &c. Q. E. D.

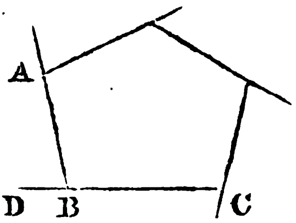
Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has ſides, by drawing ſtraight lines from a point F within the figure to each of its angles. And, by the preceding propoſition, all the angles of theſe triangles are equal to twice as many right angles as there are triangles, that is, as there are ſides of the figure; and the ſame angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles; that is [a], together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has ſides. a 2. Cor. 15. 1.



Cor. 2. All the exterior angles of any rectilineal figure, are together equal to four right angles.

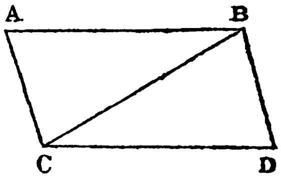
Becauſe every interior angle ABC, with its adjacent exterior ABD, is equal [b] to two right angles; therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are ſides of the figure; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal to four right angles. b 13. 1.



PROP. XXXIII. THEOR.

THE ſtraight lines which join the extremities of two equal and parallel ſtraight lines, towards the ſame parts, are alſo themſelves equal and parallel.

Let AB, CD be equal and parallel ſtraight lines, and joined towards the ſame parts by the ſtraight lines AC, BD; AC, BD are alſo equal and parallel.



Join BC; and becauſe AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal [a]: and becauſe AB is equal to CD, and BC common to the two triangles ABC, DCB, the two ſides AB, BC are equal to the two DC, CB; and the angle ABC is alſo equal to the angle BCD; therefore the a 29. 1.

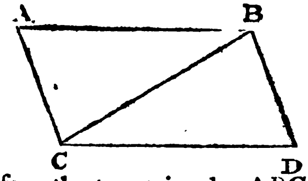
the baſe AC is equal [b] to the baſe BD, and the triangle ABC to
the triangle BCD, and the other angles to the other angles [b],
b 4. 1. each to each, to which the equal ſides are oppoſite; therefore
the angle ACB is equal to the angle CBD; and becauſe the
ſtraight line BC meets the two ſtraight lines AC, BD, and makes
the alternate angles ACB, CBD equal to one another, AC is pa-
c 27. 1. rallel [c] to BD; and it was ſhown to be equal to it. Therefore
ſtraight lines, &c. Q. E. D.

PROP. XXXIV. THEOR.

THE oppoſite ſides and angles of parallelograms are equal to one another, and the diameter biſects them, that is, divides them in two equal parts.

N. B. *A parallelogram is a four-ſided figure, of which the oppoſite ſides are parallel; and the diameter is the ſtraight line joining two of its oppoſite angles.*

Let ACDB be a parallelogram, of which BC is a diameter; the oppoſite ſides and angles of the figure are equal to one another; and the diameter BC biſects it.



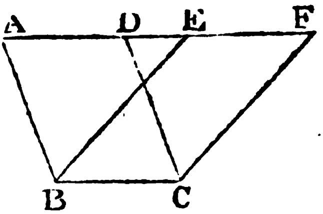
Becauſe AB is parallel to CD,
and BC meets them, the alternate
a 29. 1. angles ABC, BCD are equal [a] to
one another; and becauſe AC is
parallel to BD, and BC meets them,
the alternate angles ACB, CBD
are equal [a] to one another; wherefore the two triangles ABC,
CBD have two angles ABC, BCA in one, equal to two angles
BCD, CBD in the other, each to each, and one ſide BC com-
mon to the two triangles, which is adjacent to their equal angles;
therefore their other ſides ſhall be equal, each to each, and the
b 26. 1. third angle of the one to the third angle of the other [b], viz.
the ſide AB to the ſide CD, and AC to BD, and the angle
BAC equal to the angle BDC: And becauſe the angle ABC
is equal to the angle BCD, and the angle CBD to the angle
ACB, the whole angle ABD is equal to the whole angle ACD:
And the angle BAC has been ſhown to be equal to the angle
BDC; therefore the oppoſite ſides and angles of parallelograms
are equal to one another; alſo, their diameter biſects them; for
AB being equal to CD, and BC common, the two AB, BC are
equal to the two DC, CB, each to each; and the angle ABC is
c 4. 1. equal to the angle BCD; therefore the triangle ABC is equal [c]
to the triangle BCD, and the diameter BC divides the paralle-
logram ACDB into two equal parts. Q. E. D.

PROP. XXXV. THEOR.

PARALLELOGRAMS upon the ſame baſe, and between the ſame parallels, are equal to one another. See N.

Let the parallelograms ABCD, EBCF, be upon the ſame baſe BC, and between the ſame parallels AF, BC; the parallelogram ABCD ſhall be equal to the parallelogram EBCF.

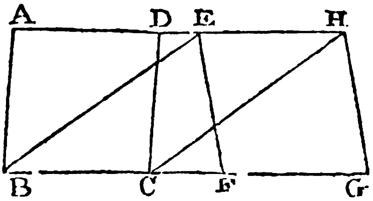
Becauſe ABCD is a parallelogram, AD is equal[a] to BC: a 34. 1.
For the ſame reaſon EF is equal to BC; therefore AD is equal
[b] to EF: Take each of theſe equals from the whole AF; and b 1. Ax.
the remainder DF is equal[c] to the remainder AE; DC alſo is c 3. Ax.
equal[a] to AB; therefore the two FD, DC are equal to the two
EA, AB, each to each; and the
exterior angle FDC is equal[d] to d 29. 1.
the interior EAB; therefore the
baſe FC is equal[e] to the baſe EB, e 4. 1.
and the triangle FDC equal to the
triangle EAB: Take each of theſe
equal triangles from the whole
figure ABCF, and the remainders
are equal[c], that is, the parallelogram ABCD is equal to the
parallelogram EBCF. Wherefore parallelograms, &c. Q. E. D.



PROP. XXXVI. THEOR.

PARALLELOGRAMS upon equal baſes, and between the ſame parallels, are equal to one another.

Let ABCD, EFGH be parallelograms upon equal baſes BC, FG, and between the ſame parallels AH, BG; the parallelogram ABCD is equal to EFGH.



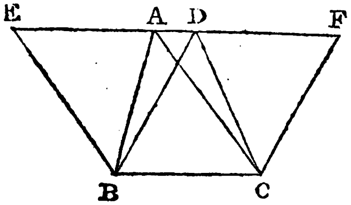
Join BE, CH; and becauſe
BC is equal to FG, and FG
to[a] EH, BC is equal to EH; and they are parallels, and joined a 34. 1.
towards the ſame parts by the ſtraight lines BE, CH: But
ſtraight lines which join equal and parallel ſtraight lines towards the ſame parts, are themſelves equal and parallel[b]; there- b 33. 1.
fore EB, CH are both equal and parallel, and EBCH is a parallelogram; and it is equal[c] to ABCD, becauſe it is upon the c 35. 1.
ſame baſe BC, and between the ſame parallels BC, AD: For
the like reaſon, the parallelogram EFGH is equal to the ſame
EBCH; therefore alſo the parallelogram ABCD is equal to
EFGH. Wherefore parallelograms, &c. Q. E. D.

Book I.

PROP. XXXVII. THEOR.

See N. TRiangles upon the ſame baſe, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DBC be upon the ſame baſe BC, and between the ſame parallels AD, BC: The triangle ABC is equal to the triangle DBC.



Produce AD both ways to the points E, F, and make EA, DF each of them equal to BC, and join BE, CF; and becauſe EA is equal and parallel to BC, the ſtraight lines BE, CA which join them are equal and parallel [a], and therefore EBCA is a parallelogram: For the ſame reaſon, DBCF is a parallelogram: and EBCA is equal [b] to DBCF, becauſe they are upon the ſame baſe BC, and between the ſame parallels BC, EF; and the triangle ABC is the half of the parallelogram EBCA, becauſe the diameter AB biſects [c] it; and the triangle DBC is the half of the parallelogram DBCF, becauſe the diameter DC biſects it: But the halves of equal things are equal [d]; therefore the triangle ABC is equal to the triangle DBC. Wherefore triangles, &c. Q. E. D.

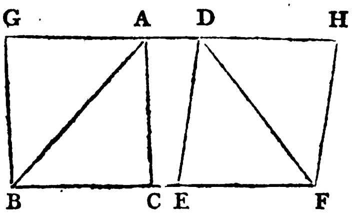
a 33. 1.
b 35. 1.
c 34. 1.
d 7. Ax.

PROP. XXXVIII. THEOR.

TRiangles upon equal baſes, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal baſes BC, EF, and between the ſame parallels BF, AD: The triangle ABC is equal to the triangle DEF.

Produce AD both ways to the points G, H, and make AG equal to BC, and DH to EF, and join BG, FH; and it may be demonſtrated, as in the preceding, that each of the figures GBCA, DEFH is a parallelogram; and they are equal [a] to one another, becauſe they are upon equal baſes BC, EF, and between the ſame parallels BF, GH: and the triangle ABC is the half [b] of the parallelogram GBCA, becauſe the diameter AB biſects it; and the triangle DEF is the half [b] of the parallelogram DEFH,



a 36. 1.
b 34. 1.

DEFH, becauſe the diameter DF biſects it: But the halves of equal things are equal [c]; therefore the triangle ABC is equal to the triangle DEF. Wherefore triangles, &c. Q. E. D. c 7. Ax.

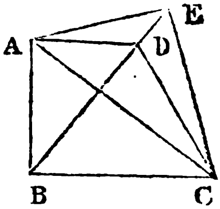
PROP. XXXIX. THEOR.

EQUAL triangles upon the ſame baſe, and upon the ſame ſide of it, are between the ſame parallels. See N.

Let the equal triangles ABC, DBC be upon the ſame baſe BC, and upon the ſame ſide of it; they are between the ſame parallels.

Join AD; AD is parallel to BC; for, if it is not, through the point A draw [a] AE parallel to BC, and becauſe AB meets the parallels BC, AE, the interior angles CBA, BAE are equal [b] to two right angles; therefore the angles DBA, BAE are leſs than two right angles; and becauſe AB meets BD, AE, and makes the interior angles DBA, BAE together leſs than two right angles, BD, AE ſhall meet one another [c]; let them meet in E, and join EC: The triangle ABC is equal [d] to the triangle EBC, becauſe it is upon the ſame baſe BC, and between the ſame parallels BC, AE: But the triangle ABC is equal to the triangle BDC; therefore alſo the triangle BDC is equal to the triangle EBC, the greater to the leſs, which is impoſſible: Therefore AE is not parallel to BC. In the ſame manner, it can be demonſtrated, that no other line but AD is parallel to BC; AD is therefore parallel to it. Wherefore equal triangles upon, &c. Q. E. D.

a 31. 1.
b 29. 1.
c 12. Ax.
d 37. 1.



Cor. Hence it is manifeſt, that the ſtraight line, which meets another ſtraight line, ſhall alſo, if produced, meet any ſtraight line parallel to that other.

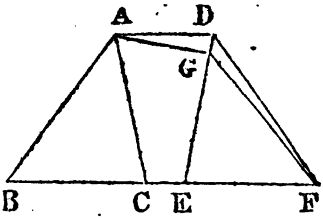
PROP. XL. THEOR.

EQUAL triangles upon equal baſes, in the ſame ſtraight line, and towards the ſame parts, are between the ſame parallels. See N.

Let the equal triangles ABC, DEF be upon equal baſes BC, EF, in the ſame ſtraight line BF, and towards the ſame parts; they are between the ſame parallels.

Join AD; AD is parallel to BC: For, if it is not, through A draw [a] AG parallel to BF; and becauſe ED meets BF, it ſhall

a 31. 1.

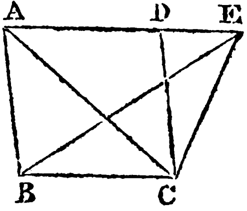


Book I. ſhall alſo meet [b] AG, which is parallel to BF; let it meet it b Cor. 39. 1. in G, and join GF: The triangle ABC is equal [c] to the triangle GEF, becauſe they are c 38. 1. upon equal baſes BC, EF, and between the ſame parallels BF, AG: But the triangle ABC is equal to the triangle DEF; therefore alſo the triangle DEF is equal to the triangle GEF, the greater to the leſs, which is impoſſible: Therefore AG is not parallel to BF: And in the ſame manner, it can be demonſtrated, that there is no other parallel to it but AD; AD is therefore parallel to BE. Wherefore equal triangles, &c. Q. E. D.

PROP. XLI. THEOR.

IF a parallelogram and a triangle be upon the ſame baſe, and between the ſame parallels; the parallelogram ſhall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the ſame baſe BC, and between the ſame parallels BC, AE; the parallelogram ABCD is double of the triangle EBC.



Join AC; then the triangle ABC a 37. 1. is equal [a] to the triangle EBC, becauſe they are upon the ſame baſe BC, and between the ſame parallels BC, AE. But the parallelogram b 34. 1. ABCD is double [d] of the triangle ABC, becauſe the diameter AC divides it into two equal parts; wherefore ABCD is alſo double of the triangle EBC. Therefore, if a parallelogram, &c. Q. E. D.

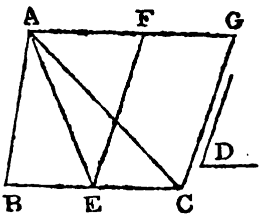
PROP. XLII. PROB.

TO deſcribe a parallelogram that ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle: It is required to deſcribe a parallelogram that ſhall be equal to the given triangle ABC, and have one of its angles equal to D.

a 10. 1. Biſect [a] BC in E, join AE, and at the point E in the ſtraight line

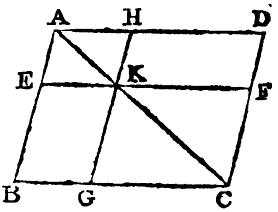
line EC make [b] the angle CEF equal to D, and through C draw b 23. 1.
CG parallel to EF, and through A
draw AG [c] parallel to BC, meeting [d] c 31. 1.
EF, CG in F, G: Therefore FECG
is a parallelogram: And becauſe BE is dCor.39.1.
equal to EC, the triangle ABE is likewiſe
equal [e] to the triangle AEC, ſince e 38. 1.
they are upon equal baſes BE, EC, and
between the ſame parallels BC, AG;
therefore the triangle ABC is double of
the triangle AEC: And the parallelogram FECG is likewiſe
double [f] of the triangle AEC, becauſe it is upon the ſame baſe, f 41. 1.
and between the ſame parallels; therefore the parallelogram
FECG is equal to the triangle ABC: and it has one of its angles
CEF equal to the given angle D: Wherefore there has been deſcribed
a parallelogram FECG equal to a given triangle ABC,
having one of its angles CEF equal to the given angle D. Which
was to be done.



PROP. XLIII. THEOR.

THE complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let ABCD be a parallelogram, of which the diameter is AC, and EH, FG the parallelograms about AC, *that is, through which* AC *paſſes*, and BK, KD the other parallelograms which make up the whole figure ABCD, which are therefore called the Complements: The complement BK is equal to the complement KD.

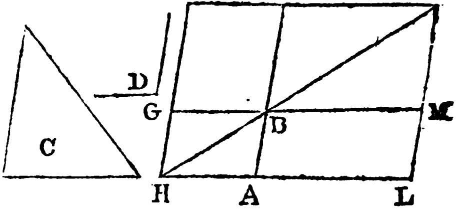


Becauſe ABCD is a parallelogram,
and AC its diameter, the triangle ABC is equal [a] to the triangle a 34. 1.
ADC: And, becauſe EKHA is a parallelogram, the diameter
of which is AK, the triangle AEK is equal to the triangle
AHK: For the ſame reaſon, the triangle KGC is equal to the triangle KFC: Then, becauſe the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC; the triangle AEK, together with the triangle KGC, is equal to the triangle AHK together with the triangle KFC: But the whole triangle ABC is equal to the whole ADC; therefore the remaining complement BK is equal to the remaining complement KD. Wherefore the complements, &c. Q. E. D.

PROP. XLIV. PROB.

TO a given ſtraight line to apply a parallelogram, which ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given ſtraight line, and C the given triangle, and D the given rectilineal angle. It is required to apply to the ſtraight line AB a parallelogram equal to the triangle C, and having an angle equal to D.



a 42. 1. Make [a] the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D, ſo that BE be in the ſame ſtraight line with AB, and produce FE, FG to K, H; and
b 31. 1. through A draw [b] AH parallel to BG or EF, meeting [c] FH in
c Cor. 39. 1. H; and join HB; it ſhall, if produced, meet [c] FK, which is
parallel to GB; let them meet in K, and through K draw KL
parallel to EA or FH, and produce HA, GB to the points L,
M: Then HLKF is a parallelogram, of which the diameter is
HK, and AG, ME are the parallelograms about HK; and LB,
e 43. 1. BF are the complements; therefore LB is equal [e] to BF: But
BF is equal to the triangle C; wherefore LB is equal to the
f 15. 1. triangle C: And becauſe the angle GBE is equal [f] to the angle
ABM, and likewiſe to the angle D; the angle ABM is equal
to the angle D: Therefore the parallelogram LB is applied to
the ſtraight line AB, is equal to the triangle C, and has the angle
ABM equal to the angle D. Which was to be done.

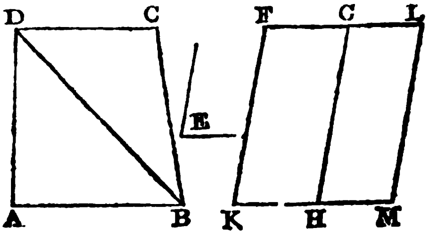
PROP. XLV. PROB.

See N. TO deſcribe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle: It is required to deſcribe a parallelogram equal to ABCD, and having an angle equal to E.

a 42. 1. Join DB, and deſcribe [a] the parallelogram FH equal to the
triangle ADB, and having the angle HKF equal to the angle
b 44. 1. E; and produce KH to M; and to the ſtraight line GH apply [b]
the

the parallelogram GM equal to the triangle DBC, having GHM
for one of its angles: and becauſe the ſtraight line GH meets
the parallels KM, FG, the alternate angles MHG, HGF are
equal [c]: add to each of theſe the angle HGL; therefore the c 29. 1.
angles MHG, HGL are equal to the angles HGF, HGL: But
MHG, HGL are equal [c] to two right angles; therefore alſo
HGF, HGL are equal to two right angles: and becauſe the two
ſtraight lines FG, GL, on oppoſite ſides of GH, make the adjacent
angles HGF, HGL equal to two right angles, FG is in
the ſame ſtraight line [d] d 14. 1.
with GL: And becauſe
KF is parallel to HG,
and HG to ML; KF
is parallel [e] to ML: e 30. 1.
And KM, FL are parallels;
wherefore KFLM
is a parallelogram; and
becauſe the triangle
ABD is equal to the parallelogram HF, and the triangle DBC
to the parallelogram GM; the whole rectilineal figure ABCD is
equal to the whole parallelogram KFLM; therefore the parallelogram
KFLM has been deſcribed equal to the given rectilineal
figure ABCD, having the angle FKM equal to the given angle
E. Which was to be done.



Cor. From this it is manifeſt, how to a given ſtraight line to
apply a parallelogram, which ſhall have an angle equal to a given
rectilineal angle, and ſhall be equal to a given rectilineal figure,
viz. by applying [b] to the given ſtraight line, a parallelogram b 44. 1.
equal to the firſt triangle ABD, and having an angle equal to
the given angle.

PROP. XLVI. PROB.

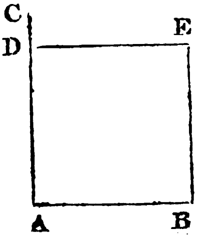
To deſcribe a ſquare upon a given ſtraight line.

Let AB be the given ſtraight line; it is required to deſcribe a ſquare upon AB.

From the point A draw [a] AC at right angles to AB; and a 11. 1.
make [b] AD equal to AB, and through the point D draw DE b 3. 1.
parallel [c] to AB, and through B draw BE parallel to AD; therefore c 31. 1.
ADEB is a parallelogram; whence AB is equal [d] to DE, d 34. 1.
and AD to BE: But BA is equal to AD; therefore the four
ſtraight lines BA, AD, DE, EB are equal to one another, and
the parallelogram ADEB is equilateral: likewiſe all its angles
are right angles; becauſe the ſtraight line AD meeting the parallels
AB, DE, the angles BAD, ADE are equal [e] to two e 29. 1.

right angles; but BAD is a right angle; therefore alſo ADE is a right angle; but the oppoſite angles of
d 34. 1. parallelograms are equal [d]; therefore each of the oppoſite angles ABE, BED is a right angle; wherefore the figure ADEB is rectangular; and it has been demonſtrated, that it is equilateral; it is therefore a ſquare, and it is deſcribed upon the given ſtraight line AB. Which was to be done.

Cor. Hence every parallelogram that has one right angle, has all its angles right angles.

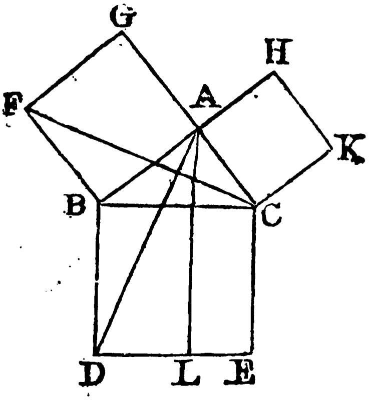


PROP. XLVII. THEOR.

In any right angled triangle, the ſquare which is deſcribed upon the ſide ſubtending the right angle, is equal to the ſquares deſcribed upon the ſides which contain the right angle.

Let ABC be a right angled triangle having the right angle BAC; the ſquare deſcribed upon the ſide BC is equal to the ſquares deſcribed upon BA, AC.

a 46. 1. On BC deſcribe [a] the ſquare BDEC, and on BA, AC the
b 31. 1. ſquares GB, HC; and through A draw [b] AL parallel to BD or
CE, and join AD, FC: then, becauſe each of the angles BAC,
c 30. Def. BAG is a right angle [c], the two ſtraight lines AC, AG upon the oppoſite ſides of AB, make with it at the point A the adjacent angles equal to two right angles; therefore CA is in
d 14. 1. the ſame ſtraight line [d] with AG; for the ſame reaſon, AB and AH are in the ſame ſtraight line: and becauſe the angle DBC is equal to the angle FBA, each of them being a right angle, add to each the angle ABC,
e 2. Ax. and the whole angle DBA is equal [e] to the whole FBC; and becauſe the two ſides AB, BD are equal to the two FB, BC, each to each, and the angle DBA equal to the angle FBC;
f 4. 1. therefore the baſe AD is equal [f] to the baſe FC, and the triangle ABD to the triangle FBC: Now the parallelogram BL is double
g 41. 1. [g] of the triangle ABD, becauſe they are upon the ſame baſe BD, and between the ſame parallels, BD, AL; and the ſquare



GB is double of the triangle FBC, becauſe theſe alſo are upon the ſame baſe FB, and between the ſame parallels FB, GC: But the doubles of equals are equal [h] to one another; therefore the parallelogram BL is equal to the ſquare GB: And in the ſame manner, by joining AE, BK, it may be demonſtrated, that the parallelogram CL is equal to the ſquare HC; therefore the whole ſquare BDEC is equal to the two ſquares GB, HC: and the ſquare BDEC is deſcribed upon the ſtraight line BC, and the ſquares GB, HC upon BA, AC: Wherefore the ſquare upon the ſide BC is equal to the ſquares upon the ſides BA, AC. Therefore, in any right angled triangle, &c. Q. E. D.

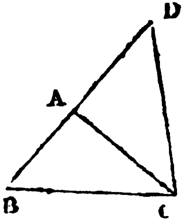
[h] 6. Ax.

PROP. XLVIII. THEOR.

IF the ſquare deſcribed upon one of the ſides of a triangle, be equal to the ſquares deſcribed upon the other two ſides of it; the angle contained by theſe two ſides is a right angle.

If the ſquare deſcribed upon BC, one of the ſides of the triangle ABC, be equal to the ſquares upon the other ſides BA, AC; the angle BAC is a right angle.

From the point A draw [a] AD at right angles to AC, and make AD equal to BA, and join DC: Then, becauſe DA is equal to AB, the ſquare of DA is equal to the ſquare of AB: To each of theſe add the ſquare of AC; therefore the ſquares of DA, AC are equal to the ſquares of BA, AC: But the ſquare of DC is equal [b] to the ſquares of DA, AC, becauſe DAC is a right angle; and the ſquare of BC, by hypotheſis, is equal to the ſquares of BA, AC; therefore the ſquare of DC is equal to the ſquare of BC; and therefore alſo the ſide DC is equal to the ſide BC. And becauſe the ſide DA is equal to AB, and AC common to the two triangles DAC, BAC, the two DA, AC are equal to the two BA, AC; and the baſe DC is alſo equal to the baſe BC; therefore the angle DAC is equal [c] to the angle BAC: But DAC is a right angle; therefore alſo BAC is a right angle. Therefore, if the ſquare, &c. Q. E. D.



[a] 11. 1.

[b] 47. 1.

[c] 8. 1.

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ELEMENTS
OF
EUCLID.

BOOK II.

DEFINITIONS.

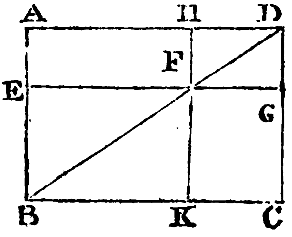
I.

Book II.

EVERY right angled parallelogram is called a rectangle, and it is ſaid to be contained by any two of the ſtraight lines which contain one of the right angles.

II.

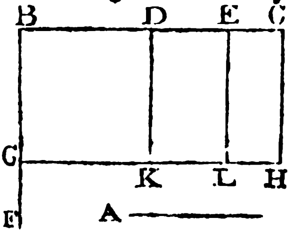
In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon. ‘Thus the parallelogram HG, together with the complements AF, FC, is the gnomon, which is more briefly expreſſed by the letters AGK, or EHC, which are at the oppoſite angles of the parallelograms which make the gnomon.’



PROP. I. THEOR.

IF there be two ſtraight lines, one of which is divided into any number of parts; the rectangle contained by the two ſtraight lines, is equal to the rectangles contained by the undivided line, and the ſeveral parts of the divided line.

Let A and BC be two ſtraight lines; and let BC be divided into any parts in the points D, E; the rectangle contained by the ſtraight lines A, BC is equal to the rectangle contained by A, BD, together with that contained by A, DE, and that contained by A, EC.

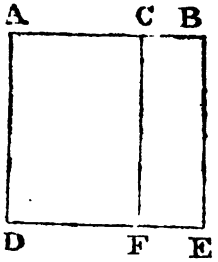


From the point B draw [a] BF at right angles to BC, and make BG equal [b] to A; and through G draw [c] GH parallel to BC; and through D, E, C draw [c] DK, EL, CH parallel to BG; then the rectangle BH is equal to the rectangles BK, DL, EH; and BH is contained by A, BC, for it is contained by GB, BC, and GB is equal to A; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained by A, DE, becauſe DK, that is, [d] BG, is equal to A; and in like manner, the rectangle EH is contained by A, EC: Therefore the rectangle contained by A, BC is equal to the ſeveral rectangles contained by A, BD, and by A, DE; and alſo by A, EC. Wherefore, if there be two ſtraight lines, &c. Q. E. D.

a 11. 1.
b 3. 1.
c 31. 1.
d 34. 1.

PROP. II. THEOR.

IF a ſtraight line be divided into any two parts, the rectangles contained by the whole, and each of the parts, are together equal to the ſquare of the whole line.



Let the ſtraight line AB be divided into any two parts in the point C; the rectangle contained by AB, BC, together with the rectangle * AB, AC, ſhall be equal to the ſquare of AB.

Upon AB deſcribe [a] the ſquare ADEB, and through C draw [b] CF, parallel to AD or BE; then AE is equal to the rectangles AF, CE; and AE is the ſquare of AB; and AF is the rectangle contained by BA, AC, for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB; therefore the rectangle contained by AB, AC, together with the rectangle

a 46. 1.
b 31. 1.

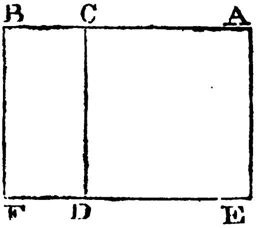
* *N. B.* To avoid repeating the word *contained* too frequently, the rectangle contained by two ſtraight lines AB, AC is ſometimes ſimply called the rectangle AB, AC.

rectangle AB, BC, is equal to the ſquare of AB. If therefore a ſtraight line, &c. Q. E. D.

PROP. III. THEOR.

If a ſtraight line be divided into any two parts, the rectangle contained by the whole, and one of the parts, is equal to the rectangle contained by the two parts, together with the ſquare of the foreſaid part.

Let the ſtraight line AB be divided into any two parts in the point C; the rectangle AB, AC is equal to the rectangle AC, CB, together with the ſquare of CA.



a 46. 1. Upon AC deſcribe [a] the ſquare
CDEA, and produce ED to F, and
b 31. 1. through B draw [b] BF parallel to CD
or AE; then the rectangle BE is equal to the rectangles BD, CE; and BE is the rectangle contained by AB, AC, for it is contained by AB, AE, of which AE is equal to AC; and BD is contained by AC, CB, for CD is equal to CA; and DA is the ſquare of AC; therefore the rectangle AB, AC is equal to the rectangle AC, CB, together with the ſquare of AC. If, therefore, a ſtraight line, &c. Q. E. D.

PROP. IV. THEOR.

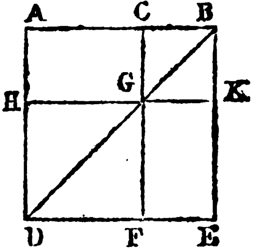
If a ſtraight line be divided into any two parts, the ſquare of the whole line is equal to the ſquares of the two parts, together with twice the rectangle contained by the parts.

Let the ſtraight line AB be divided into any two parts in C; the ſquare of AB is equal to the ſquares of AC, CB, and to twice the rectangle contained by AC, CB.

a 46. 1. Upon AB deſcribe [a] the ſquare ADEB, and join BD, and
b 31. 1. through C draw [b] CGF parallel to AD or BE, and through G
draw HK parallel to AB or DE: And becauſe CF is parallel to
c 29. 1. AD, and BD falls upon them, the exterior angle BGC is equal [c]
d 5. 1. to the interior and oppoſite angle ADB; but ADB is equal [d] to
the angle ABD, becauſe BA is equal to AD, being ſides of a
ſquare; wherefore the angle CGB is equal to the angle GBC;
e 6. 1. and therefore the ſide BC is equal [e] to the ſide CG: But CB is
f 34. 1. equal [f] alſo to GK, and CG to BK; wherefore the figure
CGKB is equilateral: It is likewiſe rectangular; for CG is parallel

rallel to BK, and CB meets them; the angles KBC, GCB are therefore equal to two right angles [c]; and KBC is a right angle; wherefore GCB is a right angle; and therefore alſo the angles [f] CGK, GKB oppoſite to theſe are right angles, and CGKB is rectangular: But it is alſo equilateral, as was demonſtrated; wherefore it is a ſquare, and it is upon the ſide CB: For the ſame reaſon, HF alſo is a ſquare, and it is upon the ſide HG, which is equal to AC: Therefore HF, CK are the ſquares of AC, CB; and becauſe the complement AG is equal [g] to the complement GE, and that AG is the rectangle contained by AC, CB, for GC is equal to CB; therefore GE is alſo equal to the rectangle AC, CB; wherefore AG, GE are equal to twice the rectangle AC, CB: And HF, CK are the ſquares of AC, CB; wherefore the four figures HF, CK, AG, GE are equal to the ſquares of AC, CB, and to twice the rectangle AC, CB: But HF, CK, AG, GE make up the whole figure ADEB, which is the ſquare of AB: Therefore the ſquare of AB is equal to the ſquares of AC, CB and twice the rectangle AC, CB. Wherefore, if a ſtraight line, &c. Q. E. D.

c 29. 1.
f 34. 1.
g 43. 1.



Cor. 1. From the demonſtration, it is manifeſt, that the parallelograms about the diameter of a ſquare, are likewiſe ſquares.

Cor. 2. Hence it is manifeſt, that the ſquares of two lines AC, CB, and twice the rectangle contained by them, are together equal to the ſquare of their ſum AB.

Cor. 3. And if AC be equal to CB or CG, the four figures AG, GE, HF, CK are ſquares, and are equal to one another; therefore AE is quadruple of CK. Conſequently the ſquare of a line is four times the ſquare of the half of that line.

PROP. V. THEOR.

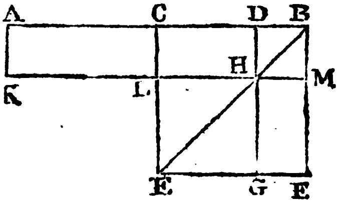
IF a ſtraight line be divided into two equal parts, and alſo into two unequal parts; the rectangle contained by the unequal parts, together with the ſquare of the line between the points of ſection, is equal to the ſquare of half the line.

Let the ſtraight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD, DB, together with the ſquare of CD, is equal to the ſquare of CB.

Upon CB deſcribe [a] the ſquare CEFB, join BE, and through D draw [b] DHG parallel to CE or BF; and through H draw

a 46. 1.
b 31. 1.

KLM parallel to CB or EF; and alſo through A draw AK parallel to CL or BM: And becauſe the complement CH is equal
c 43. 1. [c] to the complement HF, to each of theſe add DM; therefore the whole CM is equal to the
d 36. 1. whole DF; but CM is equal [d] to AL, becauſe AC is equal to CB; therefore alſo AL is equal to DF: To each of theſe add CH, and the whole AH is equal to DF and CH: But AH is the rectangle contained
e Cor. 4. 2. by AD, DB, for DH is equal [e] to DB; and DF, together with CH, is the gnomon CMG; therefore the gnomon CMG is equal to the rectangle AD, DB: To each of theſe add LG, which is equal [e] to the ſquare of CD; therefore the gnomon CMG, together with LG, is equal to the rectangle AD, DB, together with the ſquare of CD: But the gnomon CMG and LG make up the whole figure CEFB, which is the ſquare of CB: Therefore the rectangle AD, DB, together with the ſquare of CD, is equal to the ſquare of CB. Wherefore, if a ſtraight line, &c. Q. E. D.



Cor. From this propoſition it is manifeſt, that the difference of the ſquares of two unequal lines AC, CD is equal to the rectangle contained by their ſum AD and difference DB.

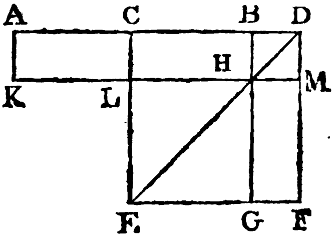
PROP. VI. THEOR.

If a ſtraight line be biſected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the ſquare of half of the line biſected, is equal to the ſquare of the ſtraight line which is made up of the half and the part produced.

Let the ſtraight line AB be biſected in C, and produced to the point D; the rectangle AD, DB, together with the ſquare of CB, is equal to the ſquare of CD.

a 46. 1. Upon CD deſcribe [a] the ſquare CEFD, join DE, and through
b 31. 1. B draw [b] BHG parallel to CE or DF, and through H draw KLM parallel to AD or EF, and alſo through A draw AK parallel to CL or DM: And becauſe AC is equal to CB, the rectangle
c 36. 1. AL is equal [c] to CH; but CH is equal [d] to HF; therefore alſo
d 43. 1. AL is equal to HF: To each of theſe add CM; therefore the whole AM is equal to the gnomon CMG: And AM is the
e Cor. 4. 2. rectangle contained by AD, DB, for DM is equal [e] to DB: Therefore the gnomon CMG is equal to the rectangle AD, DB:

DB: Add to each of theſe LG, which is equal to the ſquare of CB; therefore the rectangle AD, DB, together with the ſquare of CB, is equal to the gnomon CMG and the figure LG: But the gnomon CMG and LG make up the whole figure CEFD, which is the ſquare of CD; therefore the rectangle AD, DB, together with the ſquare of CB, is equal to the ſquare of CD. Wherefore, if a ſtraight line, &c. Q. E. D.



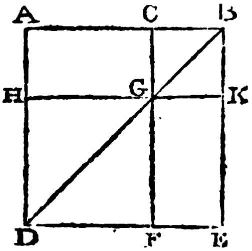
PROP. VII. THEOR.

IF a ſtraight line be divided into any two parts, the ſquares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the ſquare of the other part.

Let the ſtraight line AB be divided into any two parts in the point C; the ſquares of AB, BC are equal to twice the rectangle AB, BC, together with the ſquare of AC.

Upon AB deſcribe [a] the ſquare ADEB, and conſtruct the figure as in the preceding propoſitions: And becauſe AG is equal [b] to GE, add to each of them CK; the whole AK is therefore equal to the whole CE; therefore AK, CE are double of AK: But AK, CE are the gnomon AKF together with the ſquare CK; therefore the gnomon AKF, together with the ſquare CK, is double of AK: But twice the rectangle AB, BC is double of AK, for BK is equal [c] to BC: Therefore the gnomon AKF, together with the ſquare CK, is equal to twice the rectangle AB, BC: To each of theſe equals add HF, which is equal to the ſquare of AC; therefore the gnomon AKF, together with the ſquares CK, HF, is equal to twice the rectangle AB, BC and the ſquare of AC: But the gnomon AKF, together with the ſquares CK, HF, make up the whole figures ADEB and CK, which are the ſquares of AB and BC: Therefore the ſquares of AB and BC are equal to twice the rectangle AB, BC, together with the ſquare of AC. Wherefore, if a ſtraight line, &c. Q. E. D.

a 46. 1.
b 43. 1.
c Cor. 4. 1.



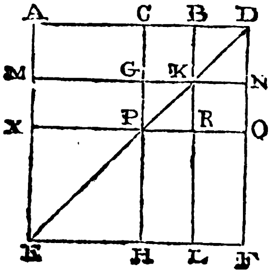
Cor. Hence it is manifeſt, that the ſquares of two lines AB, BC are equal to twice the rectangle AB, BC contained by theſe lines, together with the ſquare of their difference AC.

PROP. VIII. THEOR.

See N. IF a ſtraight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the ſquare of the other part, is equal to the ſquare of the ſtraight line which is made up of the whole and that part.

Let the ſtraight line AB be divided into any two parts in the point C; four times the rectangle AB, BC, together with the ſquare of AC, is equal to the ſquare of the ſtraight line made up of AB and BC together.

Produce AB to D, ſo that BD be equal to CB, and upon AD deſcribe the ſquare AEFD; and conſtruct two figures ſuch as in a 34. 1. the preceding. Becauſe GK is equal [a] to CB, and CB to BD; b Cor. 4. 2. GK is equal to BD: and GR, BN are ſquares [b] deſcribed upon them; therefore GR is equal to BN, and the ſide KR to the ſide BK: and becauſe BK is equal to KR, the rectangle AK is c 36. 1. equal [c] to MR: but AK is equal [d] to KF; therefore MR is equal to KF: And becauſe MP is equal d 43. 1. [d] to PL, for they are the complements of the parallelogram ML; and GR is equal to BN, the whole MR is equal to PL, BN together: and AK, KF are each of them equal to MR; therefore PL, BN together, and the three AK, KF, MR, are all equal, and ſo are quadruple of one of them AK: but PL, BN, together with AK, KF, MR, make up the gnomon AOH; therefore the gnomon AOH is quadruple of AK: and becauſe AK is the rectangle contained by AB, BC, for BK is equal to BC, four times the rectangle AB, BC is quadruple of AK: But the gnomon AOH was demonſtrated to be quadruple of AK; therefore four times the rectangle AB, BC is equal to the gnomon AOH. d Cor. 4. 2. To each of theſe add XH, which is equal [d] to the ſquare of AC: Therefore four times the rectangle AB, BC, together with the ſquare of AC, is equal to the gnomon AOH and the ſquare XH: But the gnomon AOH and XH make up the figure AEFD, which is the ſquare of AD: Therefore four times the rectangle AB, BC, together with the ſquare of AC, is equal to the ſquare of AD, that is, of AB and BC added together in one ſtraight line. Wherefore, if a ſtraight line, &c. Q. E. D.

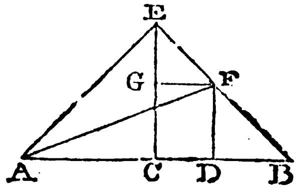


Cor. Hence, four times the rectangle contained by two lines AB, BC, together with the ſquare of their difference AC, is equal to the ſquare of their ſum AD.

PROP. IX. THEOR.

IF a ſtraight line be divided into two equal, and alſo into two unequal parts; the ſquares of the two unequal parts are together double of the ſquare of half the line, and of the ſquare of the line between the points of ſection.

Let the ſtraight line AB be divided at the point C into two equal, and at D into two unequal parts: The ſquares of AD, DB are together double of the ſquares of AC, CD.



From the point C draw [a] CE at right angles to AB, and make it equal to AC or CB, and join EA, EB; through D draw [b] DF parallel to CE, and through F draw FG parallel to AB; and join AF: Then, becauſe AC is equal to CE, the angle EAC is equal [c] to the angle AEC; and becauſe the angle ACE is a right angle, the two others AEC, EAC together make one right angle [d]; and they are equal to one another; each of them therefore is half of a right angle. For the ſame reaſon each of the angles CEB, EBC is half a right angle; and therefore the whole AEB is a right angle: And becauſe the angle GEF is half a right angle, and EGF a right angle, for it is equal [e] to the oppoſite angle ECB, the remaining angle EFG is half a right angle; therefore the angle GEF is equal to the angle EFG, and the ſide EG equal [f] to the ſide GF: Again, becauſe the angle at B is half a right angle, and FDB a right angle, for it is equal [e] to the angle ECB, the remaining angle BFD is half a right angle; therefore the angle at B is equal to the angle BFD, and the ſide DF to [f] the ſide DB. And becauſe AC is equal to CE, the ſquare of AC is equal to the ſquare of CE; therefore the ſquares of AC, CE are double of the ſquare of AC: But the ſquare of EA is equal [g] to the ſquares of AC, CE, becauſe ACE is a right angle; therefore the ſquare of EA is double of the ſquare of AC: Again, becauſe EG is equal to GF, the ſquare of EG is equal to the ſquare of GF; therefore the ſquares of EG, GF are double of the ſquare of GF: but the ſquare of EF is equal [i] to the ſquares of EG, GF; therefore the ſquare of EF is double of the ſquare GF: And GF is equal [h] to CD; therefore the ſquare of EF is double of the ſquare of

[a] 11. 1.
[b] 31. 1.
[c] 5. 1.
[d] 32. 1.
[e] 29. 1.
[f] 6. 1.
[g] 47. 1.
[i] 47. 1.
[h] 34. 1.

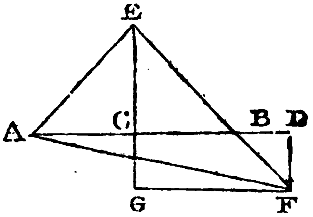
CD: But the ſquare of AE is likewiſe double of the ſquare of
AC; therefore the ſquares of AE, EF are double of the ſquares
i 47. 1. of AC, CD: And the ſquare of AF is equal [i] to the ſquares of
AE, EF, becauſe AEF is a right angle; therefore the ſquare of
AF is double of the ſquares of AC, CD: But the ſquares of
AD, DF are equal to the ſquare of AF, becauſe the angle ADF
is a right angle; therefore the ſquares of AD, DF are double of
the ſquares of AC, CD: And DF is equal to DB; therefore the
ſquares of AD, DB are double of the ſquares of AC, CD. If
therefore a ſtraight line, &c. Q. E. D.

PROP. X. THEOR.

See N. IF a ſtraight line be biſected, and produced to any point, the ſquare of the whole line thus produced, and the ſquare of the part of it produced, are together double of the ſquare of half the line biſected, and of the ſquare of the line made up of the half and the part produced.

Let the ſtraight line AB be biſected in C, and produced to the point D; the ſquares of AD, DB are double of the ſquares of AC, CD.

a 11. 1. From the point C draw [a] CE at right angles to AB: And
make it equal to AC or CB, and join AE, EB; through D draw
b 31. 1. DF parallel [b] to CE meeting [c] EB
c Cor. 39. 1. produced in F, and through F
draw FG parallel to AB meeting
[c] EC produced in G; and join
AF.



The demonſtration is the ſame with that of the preceding propoſition.

Cor. Hence, becauſe CD is half the ſum of AD, DB, for it is equal to AC, DB together, and AC is half their difference AB; it is manifeſt that the ſquares of two lines AD, DB are together double of the ſquares of CD half their ſum, and AC half their difference.

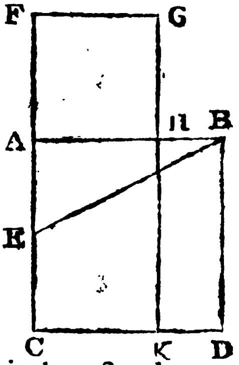
PROP. XI. PROB.

TO divide a given ſtraight line into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

Let AB be the given ſtraight line; it is required to divide it into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

Upon AB deſcribe [a] the ſquare ABDC; biſect [b] AC in E, a 46. 1.
and join BE; produce CA to F, and make [c] EF equal to EB; b 10. 1.
and upon AF deſcribe [a] the ſquare FGHA; AB is divided in c 3. 1.
H, ſo that the rectangle AB, BH is equal to the ſquare of AH.

Produce GH to K: Becauſe the ſtraight line AC is biſected in E, and produced to the point F, the rectangle CF, FA, together with the ſquare of AE, is equal [d] to the ſquare of EF: d 6. 2.
But EF is equal to EB; therefore the rectangle CF, FA, together with the ſquare of AE, is equal to the ſquare of EB: And the ſquares of BA, AE are equal [e] to the ſquare of EB, becauſe the angle EAB is a right angle; e 47. 1.
therefore the rectangle CF, FA, together with the ſquare of AE, is equal to the ſquares of BA, AE: Take away the ſquare of AE, which is common to both, therefore the remaining rectangle CF, FA is equal to the ſquare of AB: And the figure FK is the rectangle contained by CF, FA, for AF is equal to FG; and AD is the ſquare of AB; therefore FK is equal to AD: Take away the common part AK, and the remainder FH is equal to the remainder HD: And HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the ſquare of AH: Therefore the rectangle AB, BH is equal to the ſquare of AH: Wherefore the ſtraight line AB is divided in H, ſo that the rectangle AB, BH is equal to the ſquare of AH. Which was to be done.



PROP. XII. THEOR.

In obtuſe angled triangles, if a perpendicular be drawn from any of the acute angles to the oppoſite ſide produced, the ſquare of the ſide ſubtending the obtuſe angle, is greater than the ſquares of the ſides containing the obtuſe angle, by twice the rectangle contained by the ſide upon which, when produced, the perpendicular falls, and the ſtraight line intercepted, without the triangle, between the perpendicular and the obtuſe angle.

Let

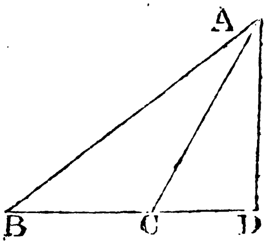
Book II.

a 12. 1.

Let ABC be an obtuſe angled triangle, having the obtuſe angle ACB, and from the point A let AD be drawn [a] perpendicular to BC produced: The ſquare of AB is greater than the ſquares of AC, CB, by twice the rectangle BC, CD.

b 4. 2. c 47. 1.

Becauſe the ſtraight line BD is divided into two parts in the point C, the ſquare of BD is equal [b] to the ſquares of BC, CD and twice the rectangle BC, CD: To each of theſe equals add the ſquare of DA; and the ſquares of BD, DA are equal to the ſquares of BC, CD, DA, and twice the rectangle BC, CD: But the ſquare of BA is equal [c] to the ſquares of BD, DA, becauſe the angle at D is a right angle; and the ſquare of CA is equal [c] to the ſquares of CD, DA: Therefore the ſquare of BA is equal to the ſquares of BC, CA, and twice the rectangle BC, CD; that is, the ſquare of BA is greater than the ſquares of BC, CA, by twice the rectangle BC, CD. Therefore, in obtuſe angled triangles, &c. Q. E. D.



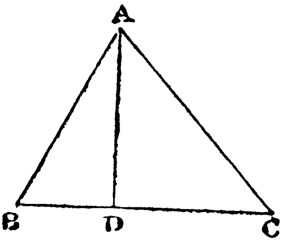
PROP. XIII. THEOR.

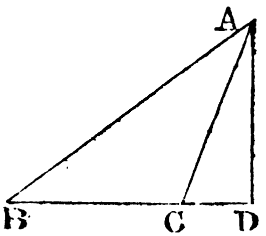
See N.

IN every triangle, the ſquare of the ſide ſubtending any of the acute angles, is leſs than the ſquares of the ſides containing that angle, by twice the rectangle contained by either of theſe ſides, and the ſtraight line intercepted between the perpendicular, let fall upon it from the oppoſite angle, and the acute angle.

a 12. 1.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the ſides containing it, let fall the perpendicular [a] AD from the oppoſite angle: The ſquare of AC, oppoſite to the angle B, is leſs than the ſquares of CB, BA by twice the rectangle CB, BD.





b 7. 2.

Becauſe the ſtraight line CB is divided into two parts in the point D, or BD in C, the ſquares of CB, BD are equal [b] to twice the rectangle contained by CB, BD, and the ſquare of DC: To each of theſe equals add the ſquare of AD; therefore the ſquares of CB, BD, DA are equal to twice the rectangle CB, BD and the ſquares of AD, DC: But the ſquare of AB is equal [c] to the ſquares of BD, DA, becauſe the angle BDA is a right angle; and the ſquare of AC is equal to the ſquares of AD, DC: Therefore the ſquares of CB, BA are equal to the ſquare of AC and twice the rectangle CB, BD; that is, the ſquare of AC alone is leſs than the ſquares of CB, BA by twice the rectangle CB, BD. Wherefore, &c. Q. E. D. c 47. 1.

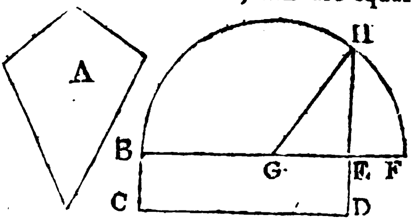
PROP. XIV. PROB.

TO deſcribe a ſquare that ſhall be equal to a given rectilineal figure.

Let A be the given rectilineal figure; it is required to deſcribe a ſquare that ſhall be equal to A.

Deſcribe [a] the rectangular parallelogram BCDE equal to the rectilineal figure A. If then the ſides of it BE, ED are equal to one another, it is a ſquare, and what was required is now done: But if they are not equal, produce one of them BE to F, and make EF equal to ED, and biſect BF in G; and from the centre G, at the diſtance GB or GF, deſcribe the ſemicircle BHF, and produce DE to H, and join GH: Therefore, becauſe the ſtraight line BF is divided into two equal parts in the point G, and into two unequal at E, the rectangle BE, EF, together with the ſquare of EG, is equal [b] to the ſquare of GF: But GF is equal to GH; therefore the rectangle BE, EF, together with the ſquare of EG, is equal to the ſquare of GH: But the ſquares of HE, EG are equal [c] to the ſquare of GH; therefore the rectangle BE, EF, together with the ſquare of EG, is equal to the ſquares of HE, EG: Take away the ſquare of EG, which is common to both; and the remaining rectangle BE, EF is equal to the ſquare of EH: But the rectangle contained by BE, EF is the parallelogram BD, becauſe EF is equal to ED; therefore BD is equal to the ſquare of EH; but BD is equal to the rectilineal figure A; therefore the rectilineal figure A is equal to the ſquare of EH: Wherefore a ſquare has been made equal to the given rectilineal figure A, viz. the ſquare deſcribed upon EH. Which was to be done.

a 45. 1.



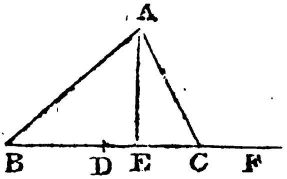
b 5. 2.

c 47. 1.

PROP. A. THEOR.

See N. IN any triangle, the difference between the ſquares of two of the ſides is equal to twice the rectangle contained by the baſe, and the ſegment of it intercepted between the perpendicular drawn to it from the oppoſite angle, and the point in which it is biſected.

Let ABC be a triangle, of which the ſide AB is greater than
a 10. 1. AC; and biſect [a] the baſe BC in D, and draw [b] AE perpendi-
b 12. 1. cular to BC; the ſquare of AB is greater than the ſquare of AC, by twice the rectangle BC. DE.



c 3. 1. Produce BC to F, and make [c] EF equal to EB: and becauſe BF is double of BE, and BC double of BD; the remainder CF is double of the remainder DE: And becauſe BF is divided into two equal parts in the point E, and into two unequal parts in the point C; the rectangle BC. CF, together with the
d 5. 2. ſquare of CE, is equal [d] to the ſquare of BE: add to theſe equals the ſquare of EA; and the rectangle BC, CF, together with the ſquares of CE, EA, is equal to the ſquares of BE, EA: But
e 1. 2. the rectangle BC, CF is double [e] of the rectangle BC, DE, be-
f 47. 1. cauſe CF is double of DE; and the ſquare of AC is equal [f] to the ſquares of CE, EA, becauſe CEA is a right angle; and the ſquare of AB is equal to the ſquares of BE, EA; therefore twice the rectangle BC, DE together with the ſquare of AC, is equal to the ſquare of AB; that is, twice the rectangle BC, DE is equal to the exceſs of the ſquare of AB above the ſquare of AC. Wherefore, &c. Q. E. D.

g 4. 1. Cor. And if AF be joined, the triangle ABF is iſoſceles [g]; hence it is manifeſt, that if any ſtraight line AC be drawn from the vertex to the baſe of an iſoſceles triangle ABF; the ſquare of the ſide AB is equal to the ſquare of the line AC, together with the rectangle BC, CF of the ſegments of the baſe.

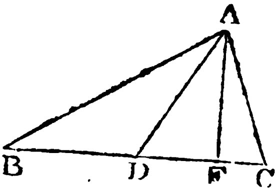
PROP. B. THEOR.

THE ſquares of two ſides of a triangle are together double of the ſquare of half the baſe, and of the ſquare of the ſtraight line drawn from the vertex to biſect the baſe.

Let ABC be a triangle; and let the baſe BC be biſected [a] in D, and join AD: the ſquares of BA, AC together are double of the ſquares of BD, DA. a 10. 1.

If the angles at D be right angles, the ſquare of BA is equal [b] to the ſquares of BD, DA, and the ſquare of AC to the ſquares of CD, DA, or of BD, DA; therefore the ſquares of BA, AC are double of the ſquares of BD, DA. b 47. 1.

But, if the angles at D be not right angles, from A draw [c] AE perpendicular to BC: and becauſe BC is biſected in D, the ſquares of BE, EC are double [d] of the ſquares of BD, DE: add to theſe equals twice the ſquare of EA; and the ſquares of BE, EC, together with twice the ſquare of EA, are double of the ſquares of BD, DE, EA: And the ſquare of DA is equal [b] to the ſquares of DE, EA, becauſe DEA is a right angle; therefore the ſquares of BE, EC, together with twice the ſquare of EA, are double of the ſquares of BD, DA: but the ſquare of BA is equal [b] to the ſquares of BE, EA, and the ſquare of AC to the ſquares of CE, EA; therefore the ſquares of BA, AC are equal to the ſquares of BE, EC, together with twice the ſquare of EA; and it has been demonſtrated, that the ſquares of BE, EC, together with twice the ſquare of EA, are double of the ſquares of BD, DA; therefore the ſquares of BA, AC are double of the ſquares of BD, DA. Wherefore, &c. Q. E. D. c 12. 1. d 9. 2.



PROP. C. THEOR.

THE ſquares of the two diameters of a parallelogram are together equal to the ſquares of its four ſides.

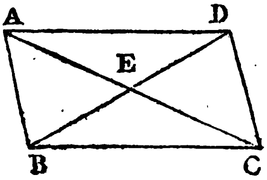
Let ABCD be a parallelogram, of which the diameters are AC, BD; the ſquares of AC, BD together are equal to the ſquares of the four ſides AB, BC, CD, DA.

Let AC, BD cut one another in E; and becauſe AC meets the parallels AD, BC, the alternate angles DAE, BCE are equal [a]: and the angle AED is equal [b] to its vertical angle BEC; therefore two angles of the triangle AED are equal to two of the triangle BEC; and the ſides AD, BC, oppoſite to equal angles, are alſo equal; therefore their other ſides are equal [c], each to each, viz. AE to EC, and BE to ED: and becauſe AE is drawn from the vertex A of the triangle BAD, to biſect the a 29. 1. b 15. 1. c 26. 1.

Book II. baſe BD; the ſquares of BA, AD are double [d] of the ſquares of BE, EA. For the ſame reaſon, the ſquares of BC, CD are double [d] of the ſquares of BE, EC, or of BE, EA, becauſe EC is equal to EA: therefore the ſquares of BA, AD, BC, CD are quadruple of the ſquares of BE, EA: but the ſquare of BD is quadruple [e] of the ſquare of BE, becauſe BD is double of BE; and the ſquare AC is quadruple [e] of the ſquare of AE; therefore the ſquares of BD, AC are quadruple of the ſquares of BE, EA: and it has been proved, that the ſquares of the four ſides are quadruple of BE, EA; therefore the ſquares of BD, AC are equal to the ſquares of the four ſides AB, BC, CD, DA. Wherefore, &c. Q. E. D.

d B. 2.

e 3 Cor. 4. 2.



Cor. Hence it is manifeſt, that the diameters of a parallelogram biſect one another.

THE ELEMENTS OF EUCLID.

BOOK III.

DEFINITIONS.

Book. III.

A.

AN arch of a circle is any part of the circumference.

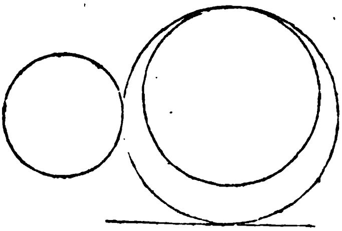
B.

A chord is any ſtraight line in a circle, terminated both ways by the circumference.

I. Omitted.

II.

A ſtraight line is ſaid to touch a circle, when it meets the circle, and being produced, does not cut it.

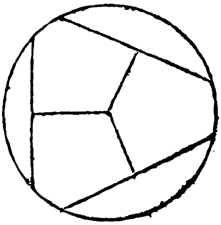


III.

Circles are ſaid to touch one another, which meet, but do not cut one another.

IV.

Straight lines are ſaid to be equally diſtant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.



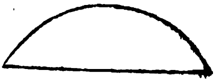
V.

And the ſtraight line on which the greater perpendicular falls, is ſaid to be farther from the centre.

VI.

VI.

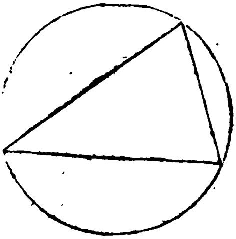
A ſegment of a circle is the figure contained by a ſtraight line, and the part of the circumference it cuts off.



VII. Omitted.

VIII.

An angle in a ſegment is the angle contained by two ſtraight lines drawn from any point in the arch of the ſegment, to the extremities of the ſtraight line which is the baſe of the ſegment.

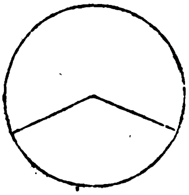


IX.

And an angle is ſaid to ſtand upon the arch intercepted between the ſtraight lines that contain the angle.

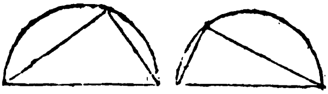
X.

The ſector of a circle is the figure contained by two ſtraight lines drawn from the centre, and the arch between them.



XI.

Similar ſegments of a circle, are thoſe in which the angles are equal, or which contain equal angles.



PROP. I. PROB.

To find the centre of a given circle.

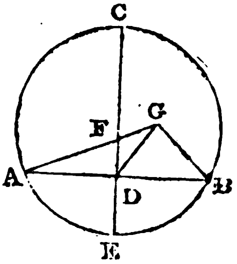
Let ABC be the given circle; it is required to find its centre.

a 10. 1. Draw within it any ſtraight line AB, and biſect [a] it in D;
b 11. 1. from the point D draw [b] DC at right angles to AB, and produce it to E, and biſect CE in F: The point F is the centre of the circle ABC.

For, if it be not, let, if poſſible, G be the centre, and join GA, GD, GB: Then, becauſe DA is equal to DB, and DG common to the two triangles ADG, BDG, the two ſides AD, DG are equal to the two BD, DG, each to each; and the baſe GA is equal to the baſe GB, becauſe they are drawn from the centre G*: Therefore the angle ADG is equal [c] to the angle GDB: But when a ſtraight line ſtanding upon another ſtraight line makes the adjacent angles equal to one another, each of the angles is a right angle [d]: Therefore the angle GDB is a right angle: But FDB is likewiſe a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the leſs, which is impoſſible: Therefore G is not the centre of the circle ABC: In the ſame manner it can be ſhewn, that no other point but F is the centre; that is, F is the centre of the circle ABC: Which was to be found.

c 8. 1.

d 10. def. 1.



Cor. From this it is manifeſt, that if in a circle a ſtraight line biſect another at right angles, the centre of the circle is in the line which biſects the other.

PROP. II. THEOR.

If any two points be taken in the circumference of a circle, the ſtraight line which joins them ſhall fall within the circle. See N.

Let ABC be a circle, and A, B any two points in the circumference; the ſtraight line drawn from A to B ſhall fall within the circle.

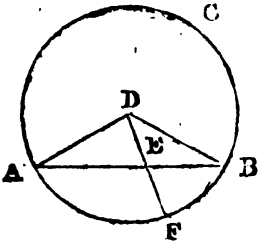
Take any point E in AB; and find [a] D the centre of the circle, and join AD, DB, DE, and let DE meet the circumference in F: Then, becauſe DA is equal to DB, the angle DAB is equal [b] to the angle DBA; and becauſe AE, a ſide of the triangle DAE, is produced to B, the angle DEB is greater [c] than the angle DAE; but DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE: But to the greater angle the greater ſide is oppoſite [d]; DB is therefore greater than DE: But DB is equal to DF; wherefore DF is greater than DE; the point E is therefore within the circle: and E is any point in the ſtraight line AB; therefore AB falls within the circle ABC. Wherefore, if any two points, &c. Q. E. D.

a 1. 3.

b 5. 1.

c 16. 1.

d 19. 1.



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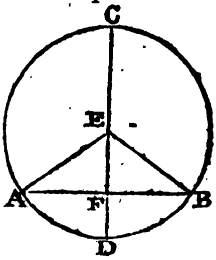
* N. B. Whenever the expreſſion "ſtraight lines from the centre," or "drawn from the centre," occurs, it is to be underſtood, that they are drawn to the circumference.

PROP. III. THEOR.

IF a ſtraight line drawn through the centre of a circle biſect a ſtraight line in it which does not paſs through the centre, it ſhall cut it at right angles; and, if it cuts it at right angles, it ſhall biſect it.

Let ABC be a circle; and let CD, a ſtraight line drawn through the centre, biſect any ſtraight line AB, which does not paſs through the centre, in the point F: It cuts it alſo at right angles.

a 1. 3. Take [a] E the centre of the circle, and join EA, EB: Then, becauſe AF is equal to FB, and FE common to the two triangles AFE, BFE, there are two ſides in the one equal to two ſides in the other, and the baſe EA is equal
b 8. 1. to EB; therefore the angle AFE is equal [b] to BFE: But when a ſtraight line ſtanding upon another makes the adjacent angles
c 10. def. 1. equal, each of them is a right [c] angle: Therefore each of the angles AFE, BFE is a right angle; wherefore the ſtraight line CD, drawn through the centre biſecting another AB that does not paſs through the centre, cuts the ſame at right angles.



But let CD cut AB at right angles; CD alſo biſects it, that is, AF is equal to FB.

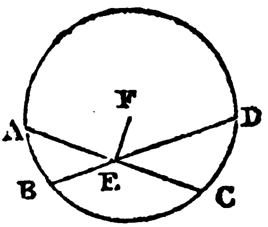
The ſame conſtruction being made, becauſe EA, EB from
d 5. 1. the centre are equal to one another, the angle EAF is equal [d] to the angle EBF; and the right angle AFE is equal to the right angle BFE: Therefore, in the two triangles EAF, EBF, there are two angles in one equal to two angles in the other, and the ſide EF, which is oppoſite to one of the equal angles
e 26. 1. in each, is common to both; therefore the other ſides are equal [e]; AF therefore is equal to FB. Wherefore, if a ſtraight line, &c. Q. E. D.

PROP. IV. THEOR.

IF in a circle two ſtraight lines cut one another which do not both paſs through the centre, they do not biſect each the other.

Let ABCD be a circle, and AC, BD two ſtraight lines in it which cut one another in the point E, and do not both paſs through the centre: AC, BD do not biſect one another.

If one of the lines paſs through the centre, it is plain that it cannot be biſected by the other which does not paſs through the centre: But, if neither of them paſs through the centre, let, if poſſible, AE be equal to EC, and BE to ED, and take [a] F the centre of the circle, and join EF: And becauſe FE, a ſtraight line through the centre, biſects another AC which does not paſs through the centre, it ſhall cut it at right [b] angles; wherefore FEA is a right angle: Again, becauſe the ſtraight line FE biſects the ſtraight line BD which does not paſs through the centre, it ſhall cut it at right [b] angles; wherefore FEB is a right angle: And FEA was ſhewn to be a right angle; therefore FEA is equal to the angle FEB, the leſs to the greater; which is impoſſible: Therefore AC, BD do not biſect one another. Wherefore, if in a circle, &c. Q. E. D.



a 1. 3.

b 3. 3.

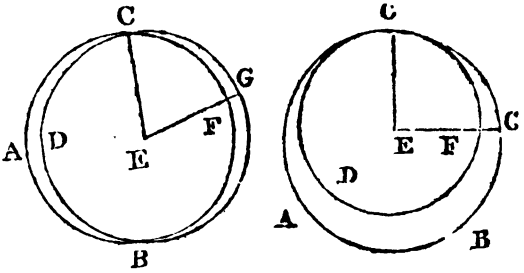
PROP. V. & VI. THEOR.

IF two circles meet one another, they ſhall not have the ſame centre.

See N.

Let the circumferences of the two circles ABC, CDG meet one another in the point C; the circles have not the ſame centre.

For, if it be poſſible, let E be their centre: Join EC, and draw any ſtraight line EFG meeting them in F and G: And becauſe E is the centre of the circle ABC, CE is equal to EF: Again, becauſe E is the centre of the circle CDG, CE is equal to EG: But CE was ſhewn to be equal to EF; therefore EF is equal to EG, the leſs to the greater; which is impoſſible: Therefore E is not the centre of the circles ABC, CDG. Wherefore, if two circles, &c. Q. E. D.

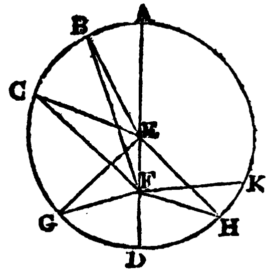


PROP. VII. THEOR.

IF any point be taken in the diameter of a circle, which is not the centre, of all the ſtraight lines which can be drawn from it to the circumference, the greateſt is that in which the centre is, and the other part of that diameter is the leaſt; and, of any others, that which is nearer to the line which paſſes through the centre is always greater than one more remote: And from the ſame point there can be drawn only two ſtraight lines that are equal to one another, one upon each ſide of the ſhorteſt line.

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the centre: Let the centre be E; of all the ſtraight lines FB, FC, FG, &c. that can be drawn from F to the circumference, FA is the greateſt, and FD, the other part of the diameter AD, is the leaſt; and of the others, FB is greater than FC, and FC than FG.

Join BE, CE, GE; and becauſe two ſides of a triangle are
a 20. 1. greater [a] than the third, BE, EF are greater than BF; but AE is equal to EB; therefore AE, EF, that is AF, is greater than BF: Again, becauſe BE is equal to CE, and FE common to the triangles BEF, CEF, the two ſides BE, EF are equal to the two CE, EF; but the angle BEF is greater than the angle CEF;
b 24. 1. therefore the baſe BF is greater [b] than the baſe FC: For the ſame reaſon, CF is greater than GF: Again, becauſe GF, FE are greater [a] than EG, and EG is equal to ED; GF, FE are greater than ED: Take away the common part FE, and the remainder GF is greater than the remainder FD: Therefore FA is the greateſt, and FD the leaſt of all the ſtraight lines from F to the circumference; and BF is greater than CF, and CF than GF.



Alſo there can be drawn only two equal ſtraight lines from the point F to the circumference, one upon each ſide of the
c 23. 1. ſhorteſt line FD: At E in EF, make [c] the angle FEH equal to GEF, and join FH: Then becauſe GE is equal to EH, and EF common to the two triangles GEF, HEF; the two ſides GE, EF are equal to the two HE, EF; and the angle GEF is equal to

to HEF; therefore the baſe FG is equal [d] to FH: But, beſides FH, no other ſtraight line can be drawn from F to the circumference equal to FG: For, if there can, let it be FK; and becauſe FK is equal to FG, and FG to FH, FK is equal to FH; that is, a line nearer to that which paſſes through the centre, is equal to one more remote; which is impoſſible. Therefore, if any point be taken, &c. Q. E. D. d 4. 1.

PROP. VIII. THEOR.

IF any point be taken without a circle, and ſtraight lines be drawn from it to the circumference, whereof one paſſes through the centre; of thoſe which fall upon the concave circumference, the greateſt is that which paſſes through the centre; and of the reſt, that which is nearer to that through the centre, is always greater than the more remote: But of thoſe which fall upon the convex circumference, the leaſt is that between the point without the circle, and the diameter; and of the reſt, that which is nearer to the leaſt is always leſs than the more remote: And only two equal ſtraight lines can be drawn from the point unto the circumference, one upon each ſide of the leaſt.

Let ABC be a circle, and D any point without it, from which let the ſtraight lines DA, DE, DF, DC be drawn to the circumference, whereof DA paſſes through the centre. Of thoſe which fall upon the concave part of the circumference AEFC, the greateſt is AD which paſſes through the centre; and the nearer to it is always greater than the more remote, viz. DE than DF, and DF than DC: But of thoſe which fall upon the convex circumference HLKG, the leaſt is DG between the point D and the diameter AG; and the nearer to it is always leſs than the more remote, viz. DK than DL, and DL than DH.

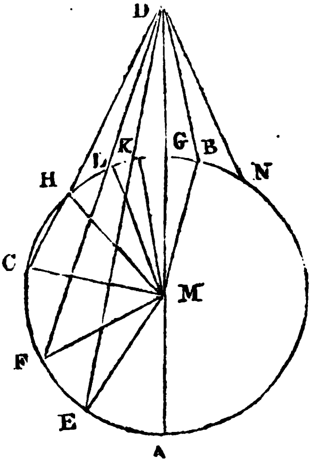
Take [a] M the centre of the circle ABC, and join ME, MF, MC, MK, ML, MH: And becauſe AM is equal to ME, add MD to each, therefore AD is equal to EM, MD; but EM, MD are greater [b] than ED; therefore alſo AD is greater than ED: Again, becauſe ME is equal to MF, and MD common to the triangles EMD, FMD; EM, MD are equal to FM, MD; but the angle EMD is greater than FMD; therefore the baſe ED is greater [c] than FD: In like manner, it may be ſhewn, that FD is greater than CD: Therefore DA is the greateſt; and

a 1. 3.

b 20. 1.

c 24. 1.

and DE greater than DF, and DF than DC: And becauſe MK,
KD are greater [b] than MD, and MK is equal to MG, the re-
b 20. 1. mainder KD is greater [d] than the
d 4. Ax. remainder GD, that is, GD is leſs
than KD: And becauſe MK, DK
are drawn to the point K within the
triangle MLD from M, D, the ex-
tremities of its ſide MD; MK, KD
c 21. 1. are leſs [c] than ML, LD, whereof
MK is equal to ML; therefore the
remainder DK is leſs than the re-
mainder DL: In like manner it may
be ſhewn, that DL is leſs than DH:
Therefore DG is the leaſt, and DK
leſs than DL, and DL than DH:
Alſo there can be drawn only two
equal ſtraight lines from the point
D to the circumference, one upon
each ſide of the leaſt: At M, in
MD, make the angle DMB equal to DMK, and join DB:
And becauſe MK is equal to MB, and MD common to the tri-
angles KMD, BMD, the two ſides KM, MD are equal to the
two BM, MD; and the angle KMD is equal to BMD; there-
f 4. 1. fore the baſe DK is equal [f] to DB: But, beſides DB, there
can be no ſtraight line drawn from D to the circumference equal
to DK: For, if there can, let it be DN; and becauſe DK is
equal to DN, and alſo to DB; therefore DB is equal to DN,
that is, the nearer to the leaſt equal to the more remote; which
is impoſſible. If, therefore, any point, &c. Q. E. D.



PROP. IX. THEOR.

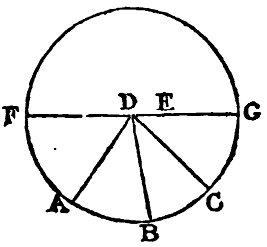
See N. IF a point be taken within a circle, from which there fall more than two equal ſtraight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal ſtraight lines, viz. DA, DB, DC, the point D is the centre of the circle.

For, if not, let E be the centre, join DE, and produce it to the circumference in F, G; then FG is a diameter of the circle ABC: And becauſe in FG, the diameter of the circle ABC, there is taken the point D which is not the centre, there can be drawn only two equal ſtraight lines from D to the circumference:

rence [a]: but DA, DB, DC are all equal; which is impoſſible; therefore E is not the centre of the circle ABC: In like manner, it may be demonſtrated, that no other point but D is the centre; D therefore is the centre. Wherefore, if a point be taken, &c. Q. E. D.

a 7. 3.



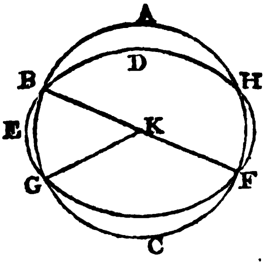
PROP. X. THEOR.

ONE circumference of a circle cannot cut another in more than two points.

If it be poſſible, let the circumference ABC cut the circumference DEF in more than two points, viz. in B, G, F; take the centre K of the circle ABC, and join KB, KG, KF: And becauſe within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal ſtraight lines KB, KG, KF, the point K is [a] the centre of the circle DEF: But K is alſo the centre of the circle ABC; therefore the ſame point is the centre of two circles that cut one another; which is impoſſible [b]. Therefore one circumference of a circle cannot cut another in more than two points. Q. E. D.

a 9. 3.

b 5. 3.



PROP. XI. THEOR.

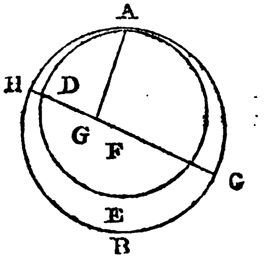
See N.

IF two circles touch each other internally, the ſtraight line which joins their centres, being produced, ſhall paſs through the point of contact.

Let the two circles ABC, ADE, of which ABC is the greater, touch each other internally in the point A, and let F be the centre of the circle ABC, and G the centre of the circle ADE: The ſtraight line which joins the centres F, G, being produced beyond G, paſſes through the point A.

For, if not, let it fall otherwiſe, if poſſible, as FGDH, and join AG: and becauſe in CH, the diameter of the circle ABC,

there is taken the point G which is not the centre, GH is the leaſt line from it to the circumference [a]; therefore AG is greater
a 7. 3. than GH: But AG is equal to GD; therefore GD is greater than GH, the leſs than the greater; which is impoſſible. Therefore the ſtraight line which joins the points F, G cannot fall otherwiſe than upon the point A, that is, it muſt paſs through it. Therefore, if two circles, &c. Q. E. D.

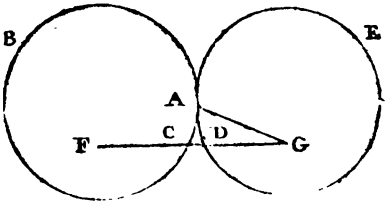


PROP. XII. THEOR.

See N. IF two circles touch each other externally, the ſtraight line which joins their centres, ſhall paſs through the point of contact.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of ADE: The ſtraight line which joins the points F, G ſhall paſs through the point of contact A.

For, if not, let it paſs otherwiſe, if poſſible, as FCDG, and join AG: and becauſe G is a point without the circle ABC, GC, which lies between it and the diameter, is the leaſt line from G to the circumfe-
a 8. 3. rence [a]; therefore GA is greater than GC: but GA is equal to GD; therefore DG is greater than GC, the leſs than the greater; which is impoſſible: Therefore the ſtraight line which joins the points F, G ſhall not paſs otherwiſe than through the point of contact A, that is, it muſt paſs through it. Therefore, if two circles, &c. Q. E. D.

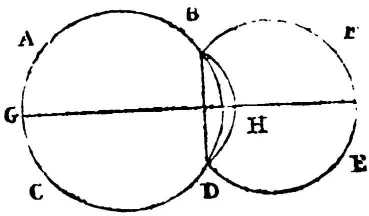


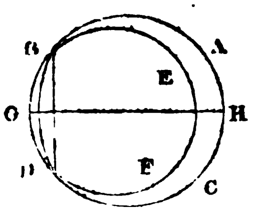
PROP. XIII. THEOR.

See N. ONE circle cannot touch another in more points than one.

For, if it be poſſible, let the circle EBF touch the circle ABC in more points than one, in the points B, D; join BD,
a 10. 11. 1. and draw [a] GH biſecting BD at right angles: Therefore, becauſe the points B, D are in the circumference of each of the circles,

circles, the ſtraight line BD falls within [b] each of them: And their centres are [c] in the ſtraight line GH which biſects BD at b 2. 3. c Cor. 1. 3.





right angles; therefore GH paſſes through the point of contact [d]; but it does not paſs through it, becauſe the points B, D are without the ſtraight line GH; which is abſurd: Therefore one circle cannot touch another in more points than one. d 11. 3. or 12. 3.

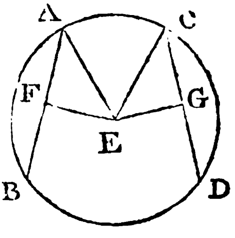
PROP. XIV. THEOR.

EQUAL ſtraight lines in a circle are equally diſtant from the centre; and thoſe which are equally diſtant from the centre, are equal to one another.

Let the ſtraight lines AB, CD, in the circle ABDC, be equal to one another; they are equally diſtant from the centre.

Take [a] E the centre of the circle ABDC, and from it draw [b] EF, EG perpendiculars to AB, CD: Then, becauſe the ſtraight line EF, paſſing through the centre, cuts AB, which does not paſs through the centre, at right angles, it alſo biſects [c] it: Wherefore AF is equal to FB, and AB double of AF. For the ſame reaſon, CD is double of CG: And AB is equal to CD; therefore AF is equal to CG: And becauſe AE is equal to EC, the ſquare of AE is equal to the ſquare of EC: But the ſquares of AF, FE are equal [d] to the ſquare of AE, becauſe AFE is a right angle; and, for the like reaſon, the ſquares of EG, GC are equal to the ſquare of EC: Therefore the ſquares of AF, FE are equal to the ſquares of CG, GE, of which the ſquare of AF is equal to the ſquare of CG, becauſe AF is equal to CG; therefore the remaining ſquare of FE is equal to the remaining ſquare of EG, and the ſtraight line FE is therefore equal to EG: But ſtraight lines in a circle are ſaid to be equally diſtant from the centre, when the perpendiculars

a 1. 3. b 12. 1. c 3. 3. d 47. 1.



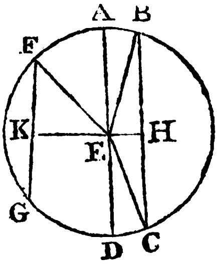
Book III. perpendiculars drawn to them from the centre are equal [e]; Therefore AB, CD are equally diſtant from the centre.

e 4. Def. 3. Next, if the ſtraight lines AB, CD be equally diſtant from the centre, that is, if FE be equal to EG; AB is equal to CD; For, the ſame conſtruction being made, it may, as before, be demonſtrated, that AB is double of AF, and CD double of CG, and that the ſquares of EF, FA are equal to the ſquares of EG, GC; of which the ſquare of FE is equal to the ſquare of EG, becauſe FE is equal to EG; therefore the remaining ſquare of AF is equal to the remaining ſquare of CG; and the ſtraight line AF is therefore equal to CG: And AB is double of AF, and CD double of CG; wherefore AB is equal to CD. Therefore equal ſtraight lines, &c. Q. E. D.

PROP. XV. THEOR.

THE diameter is the greateſt ſtraight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the leſs.

Let ABCD be a circle, of which the diameter is AD, and centre E; and let BC be nearer to the centre than FG; AD is greater than any ſtraight line BC which is not a diameter, and BC greater than FG.



a 12. 1. From the centre draw [a] EH, EK perpendiculars to BC, FG, and join EB, EC, EF; and becauſe AE is equal to EB, and ED to EC, AD is equal to

b 20. 1. EB, EC: But EB, EC are greater [b] than BC; wherefore, alſo, AD is greater than BC.

And, becauſe BC is nearer to the centre than FG, EH is

c 5. Def. 3. leſs [c] than EK: But, as was demonſtrated in the preceding, BC is double of BH, and FG double of FK, and the ſquares

d 47. 1. of EH, HB are equal [d] to the ſquares of EK, KF, of which the ſquare of EH is leſs than the ſquare of EK, becauſe EH is leſs than EK; therefore the ſquare of BH is greater than the ſquare of FK, and the ſtraight line BH greater than FK; and therefore BC is greater than FG.

Next, let BC be greater than FG; BC is nearer to the centre than FG, that is, the ſame conſtruction being made, EH is leſs than EK: Becauſe BC is greater than FG, BH likewiſe is greater than FK: And the ſquares of BH, HE are equal [c] to the ſquares of FK, KE, of which the ſquare of BH is greater than the ſquare of FK, becauſe BH is greater than FK; therefore the ſquare of EH is leſs than the ſquare of EK, and the ſtraight line EH leſs than EK. Wherefore the diameter, &c. Q. E. D.

c 5. Def. 3.

PROP. XVI. THEOR.

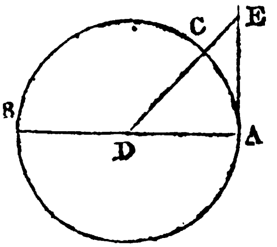
See N.

THE ſtraight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no ſtraight line can be drawn between that ſtraight line and the circumference from the extremity, ſo as not to cut the circle.

Let ABC be a circle, the centre of which is D, and the diameter AB; the ſtraight line drawn at right angles to AB from its extremity A, ſhall fall without the circle.

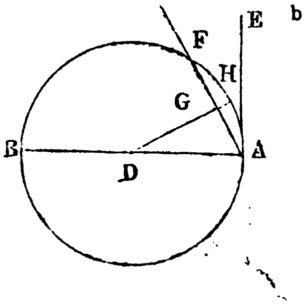
Take any point E in AE, and join DE, and let it meet the circumference in C: and becauſe the two angles DAE, AED of the triangle ADE are together leſs than two right angles [a], and that DAE is a right angle, AED muſt be leſs than a right angle: and to the greater angle the greater ſide is oppoſite [b]; therefore the ſide DE is greater than DA; but DC is equal to DA; therefore DE is greater than DC, and the point E is therefore without the circle: but E is any point whatever in AE; therefore AE falls without the circle.

a 17. 1.
b 19. 1.



And every other ſtraight line drawn through the point A cuts the circle; for, let AF be any ſtraight line, making the angle DAF leſs than the right angle DAE: and from the point D draw [c] DG perpendicular to AF, meeting the circumference in H: and becauſe DGA is a right angle, and DAG leſs than a right angle, DA is greater [b] than DG: but DH is equal to DA; therefore DH is greater than DG; therefore AF falls within the circle. Wherefore, &c. Q. E. D.

c 12. 1.
b 19. 1.



Cor. From this it is manifeſt, that the ſtraight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches only in one

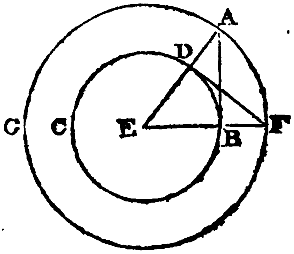
point, becauſe, if it did meet the circle in two, it would fall within it [d]. Alſo it is evident, that there can be but one ſtraight
d 2. 3. line which touches the circle in the ſame point.

PROP. XVII. PROB.

TO draw a ſtraight line from a given point, either without or in the circumference, which ſhall touch a given circle.

Firſt, let A be a given point without the given circle BCD; it is required to draw a ſtraight line from A which ſhall touch the circle.

a 1. 3 Find [a] the centre E of the circle, and join AE; and from the centre E, at the diſtance EA, deſcribe the circle AFG; from
b 11. 1. the point D draw [b] DF at right angles to EA, and join EBF, AB; AB touches the circle BCD.



Becauſe E is the centre of the circles BCD, AFG, EA is equal to EF, and ED to EB; therefore the two ſides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the baſe DF is equal to the baſe AB, and the triangle FED to the triangle AEB, and the other angles to the
c 4. 1. other angles [c]: Therefore the angle EBA is equal to the angle EDF: But EDF is a right angle, wherefore EBA is a right angle: And EB is drawn from the centre; but a ſtraight line drawn from the extremity of a diameter, at right angles to it,
dCor.16.3. touches the circle [d]: Therefore AB touches the circle; and it is drawn from the given point A. Which was to be done.

But, if the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE; DF touches the circle [d].

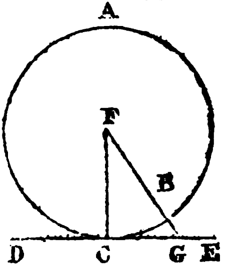
PROP. XVIII. THEOR.

IF a ſtraight line touches a circle, the ſtraight line drawn from the centre to the point of contact, ſhall be perpendicular to the line touching the circle.

Let the ſtraight line DE touch the circle ABC in the point C, take the centre F, and draw the ſtraight line FC; FC is perpendicular to DE.

For, if it be not, from the point F draw [a] FBG perpendicular to DE; and becauſe FGC is a right angle, GCF is [b] an acute angle; and to the greater angle the greater [c] ſide is oppoſite: Therefore FC is greater than FG: but FC is equal to FB; therefore FB is greater than FG, the leſs than the greater; which is impoſſible: Wherefore FG is not perpendicular to DE. In the ſame manner, it may be ſhewn, that no other is perpendicular to it beſides FC, that is, FC is perpendicular to DE. Therefore, if a ſtraight line, &c. Q. E. D.

a 12. 1.
b 17. 1.
c 19. 1.



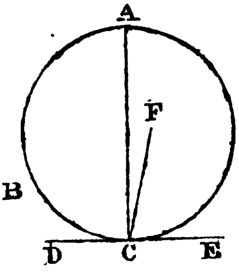
PROP. XIX. THEOR.

IF a ſtraight line touches a circle, and from the point of contact, a ſtraight line be drawn at right angles to the touching line, the centre of the circle ſhall be in that line.

Let the ſtraight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE; the centre of the circle is in CA.

For, if not, let F be the centre, if poſſible, and join CF: Becauſe DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular [a] to DE; therefore FCE is a right angle: But ACE is alſo a right angle; therefore the angle FCE is equal to the angle ACE, the leſs to the greater; which is impoſſible: Wherefore F is not the centre of the circle ABC. In the ſame manner, it may be ſhewn, that no other point which is not in CA, is the centre; that is, the centre is in CA. Therefore, if a ſtraight line, &c. Q. E. D.

a 18. 3.



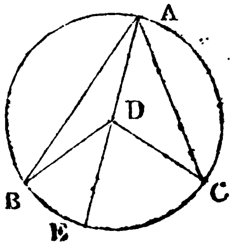
PROP. XX. THEOR.

THE angle at the centre of a circle is double of the angle at the circumference, upon the ſame baſe, that is, upon the ſame part of the circumference.

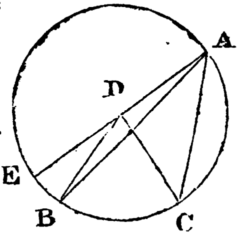
Let ABC be a circle, and BDC an angle at the centre, and BAC an angle at the circumference, which have the ſame circumference

cumference BC for their baſe; the angle BDC is double of the angle BAC.

Firſt, let D the centre of the circle be within the angle BAC, and join AD, and produce it to E: Becauſe DA is equal a 5. 1. to DB, the angle DAB is equal [a] to the angle DBA; therefore the angles DAB, DBA are double of the angle DAB; b 32. 1. but the angle BDE is equal [b] to the angles DAB, DBA; therefore alſo the angle BDE is double of the angle DAB: For the ſame reaſon, the angle EDC is double of the angle DAC: Therefore the whole angle BDC is double of the whole angle BAC.



Again, let D the centre of the circle be without the angle BAC, and join AD, and produce it to E. It may be demonſtrated, as in the firſt caſe, that the angle EDC is double of the angle EAC, and that EDB, a part of the firſt, is double of EAB, a part of the other; therefore the remaining angle BDC is double of the remaining angle BAC. Therefore the angle at the centre, &c. Q. E. D.

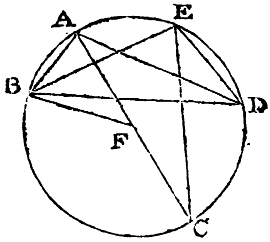


PROP. XXI, THEOR.

See N. THE angles in the ſame ſegment of a circle are equal to one another.

Let ABCD be a circle, and BAED a ſegment of it; the angles in the ſegment BAED are equal to one another.

Take F the centre of the circle, and through it draw any ſtraight line AC cutting BD, and join BA, AD; and let BED be any other angle in the ſegment BAED: the angle BAD is equal to the angle BED.



Join BF, CE: and becauſe the angle BFC is at the centre, and the angle BAC at the circumference, and that they have the ſame part of the circumference, viz. BC, for a 20. 3. their baſe; therefore the angle BFC is double [a] of the angle BAC: For the ſame reaſon, the angle BFC is double of the angle BEC: therefore the angle BAC is equal to the angle BEC. In like manner, it may be demonſtrated, that the angle CAD is equal to the angle CED: therefore the whole angle BAD is equal to the whole BED. Wherefore, the angles, &c. Q. E. D.

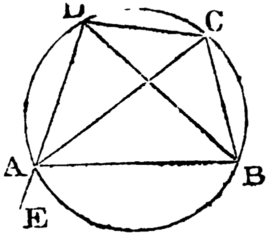
PROP.

PROP. XXII. THEOR.

THE oppoſite angles of any quadrilateral figure deſcribed in a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD; any two of its oppoſite angles are together equal to two right angles.

Join AC, BD; and becauſe the three angles of every triangle are equal [a] to two right angles, the three angles of the a 32. 1.
triangle CAB, viz. the angles CAB, ABC, BCA are equal to two right angles: But the angle CAB is equal [b] to the angle CDB, becauſe b 21. 3.
they are in the ſame ſegment BADC; and the angle ACB is equal to the angle ADB, becauſe they are in the ſame ſegment ADCB: Therefore the whole angle ADC is equal to the angles CAB, ACB: To each of theſe equals add the angle ABC; therefore the angles ABC, CAB, BCA are equal to the angles ABC, ADC: But ABC, CAB, BCA are equal to two right angles; therefore alſo the angles ABC, ADC are equal to two right angles: In the ſame manner, the angles BAD, DCB may be ſhewn to be equal to two right angles. Therefore, the oppoſite angles, &c. Q. E. D.



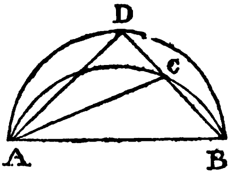
Cor. If one of the ſides DA be produced to E, the exterior angle EAB is equal to the interior oppoſite angle BCD. For EAB, BAD are equal [c] to two right angles; that is, to the c 13. 1.
two angles BCD, BAD; therefore the angle EAB is equal to BCD.

PROP. XXIII. THEOR.

UPON the ſame ſtraight line, and upon the ſame ſide of it, there cannot be two ſimilar ſegments of circles, not coinciding with one another.

If it be poſſible, let the two ſimilar ſegments of circles, viz. ACB, ADB, be upon the ſame ſide of the ſame ſtraight line AB, not coinciding with one another: Then, becauſe the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point [a]: One of the ſegments muſt a 10. 3.
therefore fall within the other; let ACB fall within ADB, and draw

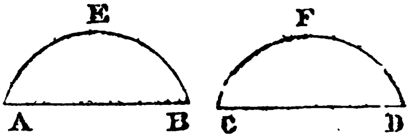
Book III. draw the ſtraight line BCD, and join CA, DA: And becauſe
the ſegment ACB is ſimilar to the ſegment ADB, and that ſimi-
b 11. def. 3. lar ſegments of circles contain [b]
equal angles; the angle ACB is
equal to the angle ADB, the exte-
rior to the interior, which is im-
c 16. 1. poſſible [c]. Therefore, there cannot
be two ſimilar ſegments of a circle
upon the ſame ſide of the ſame line,
which do not coincide. Q. E. D.



PROP. XXIV. THEOR.

SIMILAR ſegments of circles upon equal ſtraight lines, are equal to one another.

Let AEB, CFD be ſimilar ſegments of circles upon the equal ſtraight lines AB, CD; the ſegment AEB is equal to the ſegment CFD.



For, if the ſegment AEB be applied to the ſegment CFD, ſo as the point A be on C, and the ſtraight line AB
upon CD, the point B ſhall coincide with the point D, becauſe
AB is equal to CD: Therefore the ſtraight line AB coinciding
a 23. 3. with CD, the ſegment AEB muſt [a] coincide with the ſegment
CFD, and therefore is equal to it. Wherefore ſimilar ſegments,
&c. Q. E. D.

PROP. XXV. PROB.

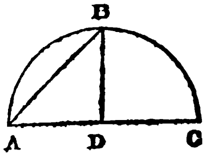
A SEGMENT of a circle being given, to deſcribe the circle of which it is the ſegment.

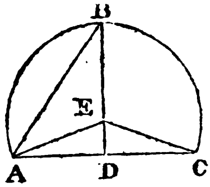
Let ABC be the given ſegment of a circle; it is required to deſcribe the circle of which it is the ſegment.

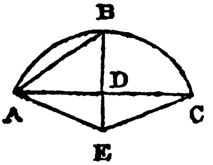
a 10. 1. Biſect [a] AC in D, and from the point D draw [b] DB at right
b 11. 1. angles to AC, and join AB. Firſt, let the angles ABD, BAD,
c 6. 1. be equal; then BD is equal [c] to DA, and therefore to DC;
and becauſe DA, DB, DC are all equal, D is the centre of the
d 9. 3. circle [d]: From the centre D, at the diſtance of any of the three
DA, DB, DC, deſcribe a circle; this ſhall paſs through the
other points, and ſhall be the circle of which ABC is a ſegment:
And becauſe the centre D is in AC, the ſegment ABC is a
ſemicircle. But if the angles ABD, BAD are not equal, at A,

in AB make [e] the angle BAE equal to ABD, and produce BD, if neceſſary, to E, and join EC: And becauſe the angle ABE is equal to BAE, BE is equal [c] to EA: And becauſe AD is equal to DC, and DE common to the triangles ADE, CDE, the two

e 23. 1.
c 6. 1.







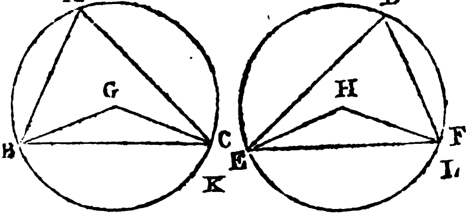
ſides AD, DE are equal to the two CD, DE, each to each; and the angle ADE is equal to CDE, for each of them is a right angle; therefore the baſe AE is equal [f] to EC: But AE was ſhewn to be equal to EB; wherefore alſo BE is equal to EC: And the three AE, EB, EC are therefore equal; wherefore [d] E is the centre of the circle. From the centre E, at the diſtance of any of the three AE, EB, EC, deſcribe a circle, this ſhall paſs through the other points, and be the circle of which ABC is a ſegment: And it is evident, that if the angle ABD be greater than BAD, the centre E falls without the ſegment ABC, which therefore is leſs than a ſemicircle: But if the angle ABD be leſs than BAD, the centre E falls within the ſegment ABC, which is therefore greater than a ſemicircle: Wherefore a ſegment of a circle being given, the circle is deſcribed of which it is a ſegment. Which was to be done.

f 4. 1.
d 9. 3.

PROP. XXVI. THEOR.

IN equal circles, equal angles ſtand upon equal arches, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and BGC, EHF equal angles at their centres, and BAC, EDF at their circumferences: The arch BKC is equal to the arch ELF.



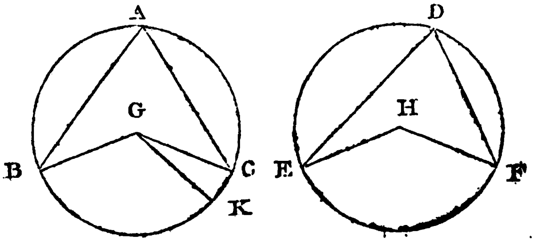
Join

Book III. Join BC, EF; and becauſe the circles ABC, DEF are equal, the ſtraight lines drawn from their centres are equal: Therefore the two ſides BG, GC are equal to the two EH, HF; and the angle at G is equal to the angle at H; therefore the
a 4. 1. baſe BC is equal [a] to the baſe EF: And becauſe the angle at A
b 11.def.3. is equal to the angle at D, the ſegment BAC is ſimilar [b] to the ſegment EDF; and they are upon equal ſtraight lines BC, EF; but ſimilar ſegments of circles upon equal ſtraight lines are
c 24. 3. equal [c] to one another; therefore the ſegment BAC is equal to the ſegment EDF: But the whole circle ABC is equal to the whole DEF; therefore the remaining ſegment BKC is equal to the remaining ſegment ELF, and the arch BKC to the arch ELF. Wherefore, in equal circles, &c. Q. E. D.

PROP. XXVII. THEOR.

IN equal circles, the angles which ſtand upon equal arches, are equal to one another, whether they be at the centres or circumferences.

Let the angles BGC, EHF at the centres, and BAC, EDF at the circumferences of the equal circles ABC, DEF, ſtand upon the equal arches BC, EF: The angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF. For, if the angle BGC be not equal to the angle EHF, at the point G,



a 23. 1. in the ſtraight line BG, make [a] the angle BGK equal to the
b 26. 3. angle EHF; but equal angles ſtand upon equal arches [b], when they are at the centre; therefore the arch BK is equal to EF: But EF is equal to BC; therefore alſo BK is equal to BC, the leſs to the greater; which is impoſſible: Therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it:
c 20. 3. And the angle at A is half [c] of the angle BGC, and the angle at D half of the angle EHF: Therefore the angle at A is equal to the angle at D. Wherefore, in equal circles, &c. Q. E. D.

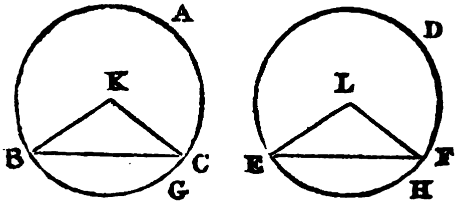
PROP.

PROP. XXVIII. THEOR.

IN equal circles, equal ſtraight lines cut off equal arches, the greater equal to the greater, and the leſs to the leſs.

Let ABC, DEF be equal circles, and BC, EF equal ſtraight lines in them, which cut off the two greater arches BAC, EDF, and the two leſs BGC, EHF: The greater BAC is equal to the greater EDF, and the leſs BGC to the leſs EHF.

Take [a] K, L, the centres of the circles, and join BK, KC, EL, LF: And becauſe the circles are equal, the ſtraight lines from their centres are equal; therefore BK, KC are equal to EL, LF; and the baſe BC is equal to the baſe EF; therefore the angle BKC is equal [b] to the angle ELF: But equal angles ſtand upon equal [c] arches, when they are at the centres; therefore the arch BGC is equal to EHF. But the whole circle ABC is equal to the whole EDF; the remaining part therefore of the circumference, viz. BAC, is equal to the remaining part EDF. Therefore, in equal circles, &c. Q. E. D.



a 1. 3.
b 8. 1.
c 26. 3.

PROP. XXIX. THEOR.

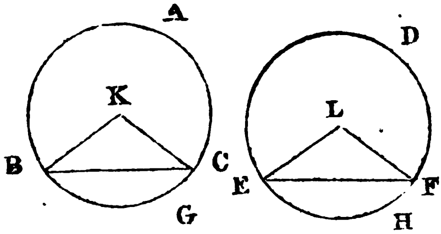
IN equal circles, equal arches are ſubtended by equal ſtraight lines.

Let ABC, DEF be equal circles, and let the arches BGC, EHF alſo be equal; and join BC, EF: The ſtraight line BC is equal to the ſtraight line EF.

Take [a] K, L, the centres of the circles, and join BK, KC, EL, LF: And becauſe the arch BGC is equal to EHF, the angle BKC is equal [b] to the angle ELF: And becauſe the circles ABC, DEF are equal, the ſtraight lines from their centres are equal: Therefore BK, KC are equal to EL, LF,

a 1. 3.
b 27. 3.

LF, and they contain equal angles: Therefore the baſe BC



c 4. 1. is equal [c] to the baſe EF. Therefore, in equal circles, &c. Q. E. D.

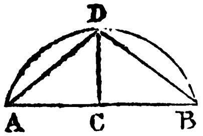
PROP. XXX. PROB.

TO biſect a given arch, *that is, to divide it into two equal parts.*

Let ADB be the given arch; it is required to biſect it.

a 10. 1. Join AB, and biſect [a] it in C; from the point C draw [b] CD
b 11. 1. at right angles to AB, and join AD, DB: The arch ADB is biſected in the point D.

Becauſe AC is equal to CB, and CD common to the triangles ACD, BCD, the two ſides AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, becauſe each of them is a right angle;
c 4. 1. therefore the baſe AD is equal [c] to the baſe BD: But equal ſtraight lines cut off
d 28. 3. equal [d] arches; the greater equal to the greater, and the leſs to the leſs, and AD, DB are each of them leſs than a ſemicircle;
d Cor. 1. 3. becauſe DC paſſes through the centre [d]: Wherefore the arch AD is equal to DB: Therefore the given arch is biſected in D. Which was to be done.



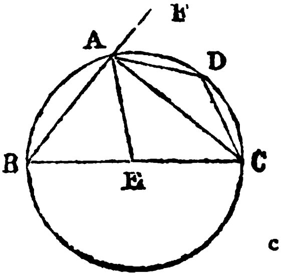
Cor. Hence it is manifeſt, that the ſtraight line drawn from the centre to biſect a chord, biſects alſo the arch cut off by it; or if it biſect the arch, it biſects alſo the chord.

PROP. XXXI. THEOR.

IN a circle, the angle in a ſemicircle is a right angle; but the angle in a ſegment greater than a ſemicircle is leſs than a right angle; and the angle in a ſegment leſs than a ſemicircle, is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and centre E; and draw CA, dividing the circle into the ſegments ABC, ADC, and join BA, AD, DC; the angle in the ſemicircle BAC is a right angle; and the angle in the ſegment ABC, which is greater than a ſemicircle, is leſs than a right angle; and the angle in the ſegment ADC, which is leſs than a ſemicircle, is greater than a right angle.

Join AE, and produce BA to F; and becauſe BE is equal to EA, the angle EAB is equal [a] to EBA; alſo, becauſe AE is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB: But FAC, the exterior angle of the triangle ABC, is equal [b] to the two angles ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a right [c] angle: Wherefore the angle BAC in a ſemicircle is a right angle.



a 5. 1.
b 32. 1.
c 10. def. 1.

And becauſe the two angles ABC, BAC of the triangle ABC are together leſs [d] than two right angles, and that BAC is a right angle, ABC muſt be leſs than a right angle; and therefore the angle in a ſegment ABC, greater than a ſemicircle, is leſs than a right angle.

d 17. 1.

And becauſe ABCD is a quadrilateral figure in a circle, any two of its oppoſite angles are equal [e] to two right angles; therefore the angles ABC, ADC are equal to two right angles; and ABC is leſs than a right angle; wherefore the other ADC is greater than a right angle.

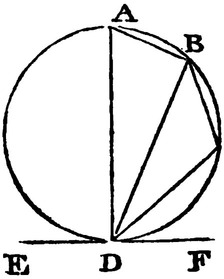
e 22. 3.

PROP. XXXII. THEOR.

IF a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn, cutting the circle, the angles made by this line, with the line touching the circle, ſhall be equal to the angles which are in the alternate ſegments of the circle.

Let the ſtraight line EF touch the circle ABCD in D, and from the point D, let the ſtraight line BD be drawn cutting the circle: The angles which BD makes with the touching line EF, ſhall be equal to the angles in the alternate ſegments of the circle; that is, the angle FDB is equal to the angle which is in the ſegment DAB, and the angle BDE to the angle in the ſegment BCD.

From the point D draw [a] DA at right angles to EF, and
take any point C in the arch BD, and join AB, DC, CB; and
a 11. 1. becauſe the ſtraight line EF touches the circle ABCD in the
point D, and DA is drawn at right
angles to the touching line from the
point of contact D, the centre of the
b 19. 3. circle is [b] in DA; therefore the angle
c 31. 3. ABD in a ſemicircle is a right [c] angle,
and conſequently the other two angles
d 32. 1. BAD, ADB are equal [d] to a right
angle: But ADF is likewiſe a right
angle; therefore the angle ADF is
equal to the angles BAD, ADB: Take
from theſe equals the common angle ADB; therefore the remaining angle BDF is equal to the angle BAD, which is in the
alternate ſegment of the circle; and becauſe ABCD is a quadrilateral figure in a circle, the oppoſite angles BAD, BCD are
e 22. 3. equal [e] to two right angles; therefore the angles BDF, BDE,
f 13. 1. being likewiſe equal [f] to two right angles, are equal to the
angles BAD, BCD; and BDF has been proved equal to BAD: Therefore the remaining angle BDE is equal to the angle BCD in the alternate ſegment of the circle. Wherefore, if a ſtraight line, &c. Q. E. D.

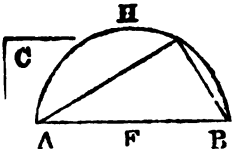


PROP. XXXIII. PROB.

UPON a given ſtraight line, to deſcribe a ſegment of a circle, containing an angle equal to a given rectilineal angle.

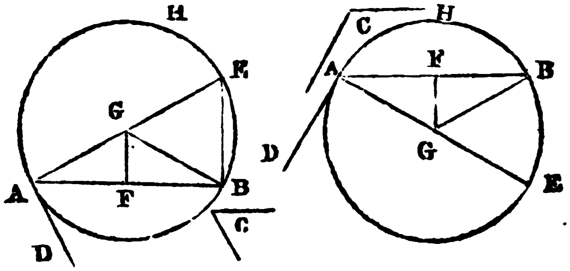
Let AB be the given ſtraight line, and the angle at C the given angle; it is required to deſcribe upon the given ſtraight line AB, a ſegment of a circle, containing an angle equal to the angle C.

Firſt, let the angle at C be a right
a 10. 1. angle, and biſect [a] AB in F, and from
the centre F, at the diſtance FB, deſcribe the ſemicircle AHB; therefore
b 31. 3. the angle AHB in a ſemicircle is [b] equal
to the right angle at C.



But, if the angle C be not a right angle, at the point A, in
c 23. 1. the ſtraight line AB, make [c] the angle BAD equal to the angle
d 11. 1. C, and from the point A, draw [d] AE at right angles to AD;
biſect [a] AB in F, and from F draw [d] FG at right angles to AB, and join GB: And becauſe AF is equal to FB, and FG common to the triangles AFG, BFG, the two ſides AF, FG are equal to

to the two BF, FG; and the angle AFG is equal to the angle BFG; therefore the baſe AG is equal [e] to the baſe GB; and the circle deſcribed from the centre G, at the diſtance GA, ſhall paſs through the point B; let this be the circle AHB: And be- e 4. 1.

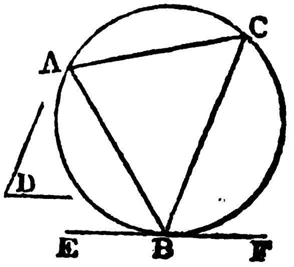


cauſe from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, therefore AD [f] touches the circle; and becauſe AB drawn from the point of contact A cuts the circle, the angle DAB is equal to the angle in the alternate ſegment AHB [g]: But the angle DAB is equal to the angle C, therefore alſo the angle C is equal to the angle in the ſegment AHB: Wherefore, upon the given ſtraight line AB the ſegment AHB of a circle is deſcribed, which contains an angle equal to the given angle at C. Which was to be done. f Cor.16.3. g 32. 3.

PROP. XXXIV. PROB.

TO cut off a ſegment from a given circle which ſhall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given angle; it is required to cut off a ſegment from the circle ABC that ſhall contain an angle equal to the angle D.



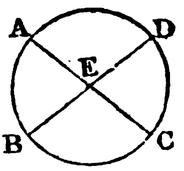
Draw [a] the ſtraight line EF, touching the circle ABC in the point B, and at the point B, in the ſtraight line BF, make [b] the angle FBC equal to the angle D: Therefore, becauſe the ſtraight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal [c] to the angle in the alternate ſegment BAC of the circle: But a 17. 3. b 23. 1. c 32. 3.

But the angle FBC is equal to the angle D; therefore the angle in the ſegment BAC is equal to the angle D: Wherefore the ſegment BAC is cut off from the given circle ABC, containing an angle equal to the given angle D. Which was to be done.

PROP. XXXV. THEOR.

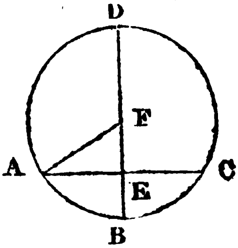
IF two ſtraight lines within a circle cut one another, the rectangle contained by the ſegments of one of them, is equal to the rectangle contained by the ſegments of the other.

Let the two ſtraight lines AC, BD, within the circle ABCD, cut one another in the point E; the rectangle contained by AE, EC is equal to the rectangle contained by BE, ED.



If AC, BD paſs each of them through the centre, ſo that E is the centre; it is evident, that AE, EC, BE, ED, being all equal, the rectangle AE, EC is likewiſe equal to the rectangle BE, ED.

But let one of them BD paſs through the centre, and cut the other AC, which does not paſs through the centre, at right angles, in the point E: Then, if BD be biſected in F, F is the centre, join AF: And becauſe BD, which paſſes through the centre, cuts AC, which does not paſs through the centre, at right angles in E, a 3. 3. AE, EC are equal [a] to one another: And becauſe BD is cut into two equal parts in F, and into two unequal in E, the rectangle BE, ED, together with the ſquare b 5. 2. of EF, is equal [b] to the ſquare of FB; that is, to the ſquare of FA; but the c 47. 1. ſquares of AE, EF are equal [c] to the ſquare of FA; therefore the rectangle BE, ED, together with the ſquare of EF, is equal to the ſquares of AE, EF: Take away the common ſquare of EF, and the remaining rectangle BE, ED is equal to the remaining ſquare of AE; that is, to the rectangle AE, EC.

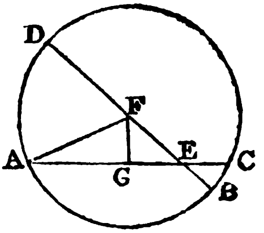


Next, let BD, which paſſes through the centre, cut the other AC, which does not paſs through the centre, in E, but not at right angles: Then, as before, if BD be biſected in F, F is the d 12. 1. centre: Join AF, and from F draw [d] FG perpendicular to AC; a 3. 3. therefore AG is equal [a] to GC; wherefore the rectangle AE, b 5. 2. EC, together with the ſquare of EG, is equal [b] to the ſquare of AG;

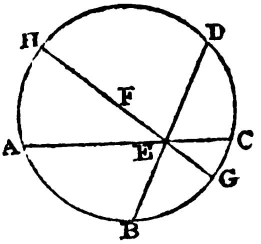
AG: To each of theſe equals add the ſquare of GF; therefore the rectangle AE, EC, together with the ſquares of EG, GF, is equal to the ſquares of AG, GF: But the ſquares of EG, GF are equal [c] to the ſquare of EF; and the ſquares of AG, GF are equal to the ſquare of AF: Therefore the rectangle AE, EC, together with the ſquare of EF, is equal to the ſquare of AF; that is, to the ſquare of FB: But the ſquare of FB is equal [b] to the rectangle BE, ED, together with the ſquare of EF; therefore the rectangle AE, EC, together with the ſquare of EF, is equal to the rectangle BE, ED, together with the ſquare of EF: Take away the common ſquare of EF, and the remaining rectangle AE, EC is therefore equal to the remaining rectangle BE, ED.

c 47. 1.

b 5. 2.



Laſtly, let neither of the ſtraight lines AC, BD paſs through the centre: Take the centre F, and join FE, and produce it to the circumference in G, H: And becauſe the rectangle AE, EC is equal, as has been ſhown, to the rectangle GE, EH; and for the ſame reaſon, the rectangle BE, ED is equal to the ſame rectangle GE, EH; therefore the rectangle AE, EC is equal to the rectangle BE, ED. Wherefore, if two ſtraight lines, &c. Q. E. D.



PROP. XXXVI. THEOR.

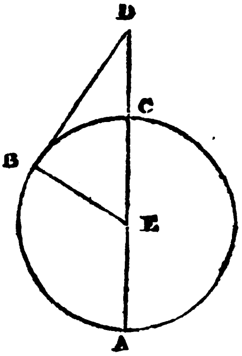
IF from any point without a circle, two ſtraight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, ſhall be equal to the ſquare of the line which touches it.

Let D be any point without the circle ABC, and DCA, DB two ſtraight lines drawn from it, of which DCA cuts the circle, and DB touches the ſame: The rectangle AD, DC is equal to the ſquare of DB.

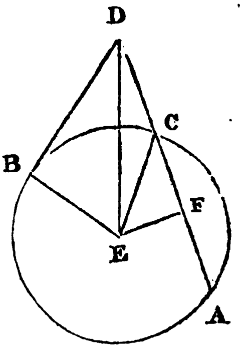
Either DCA paſſes through the centre, or it does not; firſt, let it paſs through the centre E, and join EB; therefore the angle EBD is a right [a] angle: And becauſe the ſtraight line AC is

a 18. 3.

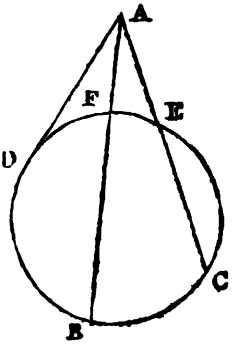
is biſected in E, and produced to the point D, the rectangle AD,
DC, together with the ſquare of EC, is equal[b] to the ſquare
b 6. 2. of ED, and CE is equal to EB: Therefore
the rectangle AD, DC, together
with the ſquare of EB, is equal to the
ſquare of ED: But the ſquare of ED
c 47. 1. is equal[c] to the ſquares of EB, BD, becauſe
EBD is a right angle: Therefore
the rectangle AD, DC, together with
the ſquare of EB, is equal to the ſquares
of EB, BD: Take away the common
ſquare of EB; therefore the remaining
rectangle AD, DC is equal to the ſquare
of DB.



But if DCA does not paſs through
d 1. 3. the centre of the circle ABC, take[d] the centre E, and draw EF
e 12. 1. perpendicular[e] to AC, and join EB, EC, ED: And becauſe
EF, which paſſes through the centre, cuts AC, which does not
paſs through the centre, at right angles,
f 3. 3. it ſhall biſect[f] it; therefore AF is equal
to FC: And becauſe AC is biſected in F,
and produced to D, the rectangle AD,
DC, together with the ſquare of FC, is
equal[b] to the ſquare of FD: To each of
theſe equals add the ſquare of FE; therefore
the rectangle AD, DC, together
with the ſquares of CF, FE, is equal to
the ſquares of DF, FE: But the ſquare
of ED is equal[c] to the ſquares of DF,
FE, becauſe EFD is a right angle; and
the ſquare of EC is equal to the ſquares
of CF, FE; therefore the rectangle AD, DC, together with
the ſquare of EC, is equal to the ſquare of ED: And CE
is equal to EB; therefore the rectangle AD, DC, together
with the ſquare of EB, is equal to the ſquare of ED: But
c 47. 1. the ſquares of EB, BD are equal to the ſquare[c] of ED, becauſe
EBD is a right angle; therefore the rectangle AD, DC,
together with the ſquare of EB, is equal to the ſquares of EB,
BD: Take away the common ſquare of EB; therefore the remaining
rectangle AD, DC is equal to the ſquare of DB.
Wherefore, if from any point, &c. Q. E. D.



Cor. If from any point without a circle, there be drawn two ſtraight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle BA, AF to the rectangle CA, AE: For each of them is equal to the ſquare of the ſtraight line AD which touches the circle.

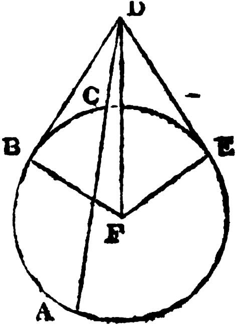


PROP. XXXVII. THEOR.

IF from a point without a circle there be drawn two ſtraight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the ſquare of the line which meets it; the line which meets ſhall touch the circle.

Let any point D be taken without the circle ABC, and from it let two ſtraight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; if the rectangle AD, DC be equal to the ſquare of DB; DB touches the circle.

Draw[a] the ſtraight line DE touching the circle ABC, find a 17. 3.
its centre F, and join FE, FB, FD; then FED is a right[b] b 18. 3.
angle: And becauſe DE touches the circle ABC, and DCA cuts
it, the rectangle AD, DC is equal[c] to the ſquare of DE: But c 36. 3.
the rectangle AD, DC is, by hypotheſis, equal to the ſquare of DB: Therefore the ſquare of DE is equal to the ſquare of DB; and the ſtraight line DE equal to the ſtraight line DB: And FE is equal to FB, wherefore DE, EF are equal to DB, BF; and the baſe FD is common to the two triangles DEF, DBF; therefore the angle DEF is equal[d] to the angle d 8. 1.
DBF; but DEF is a right angle, therefore alſo DBF is a right angle: And FB, if produced, is a diameter, and the ſtraight line which is drawn at right



angles to a diameter, from the extremity of it, touches [e] the circle: Therefore DB touches the circle ABC. Wherefore, if from a point, &c. Q. E. D.

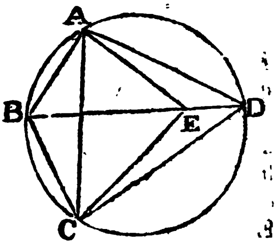
e 16. 3.

Cor. Hence ſtraight lines DB, DE, drawn to touch a circle ABC from the ſame point D without it, are equal to one another.

PROP. A. THEOR.

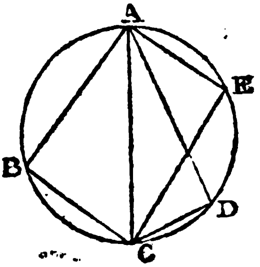
If the oppoſite angles of a quadrilateral figure be together equal to two right angles, and the circumference of a circle paſs through three of theſe angles, it ſhall alſo paſs through the fourth.

Let ABCD be a quadrilateral figure, of which the oppoſite angles are equal to two right angles; and let the circumference of a circle paſs through the three points A, B, C; it ſhall alſo paſs through the point D.



Join AC, BD: and if the circumference ABC do not paſs through D, let it, if poſſible, meet BD in the point E; and join AE, EC; therefore the
a 21. 1. angle AEC is unequal [a] to the angle ADC: but becauſe ABCE is a quadrilateral figure in a circle, the oppoſite
b 22. 3. angles ABC, AEC are equal [b] to two right angles: and the two ABC, ADC are alſo equal to two right angles; therefore the angles ABC, AEC are equal to the angles ABC, ADC: take away the common angle ABC, and the remaining angle AEC is equal to the remaining angle ADC: and they are alſo unequal, which is impoſſible: Therefore the circumference ABC cannot but paſs through D; that is, it paſſes through it. Wherefore, &c. Q. E. D.

Cor. 1. And if two triangles ACD, ACE, on the ſame baſe AC, have their vertical angles at D, E equal to one another; the circumference which paſſes through the points A, C, D, ſhall alſo paſs through E.



In the circumference take any point B on the other ſide of AC, and join AB, BC; therefore ABC, ADC are
b 22. 3. equal [b] to two right angles: but ADC is equal to AEC; therefore ABC, AEC are equal to two right angles; wherefore the circumference ABC paſſes through the point E.

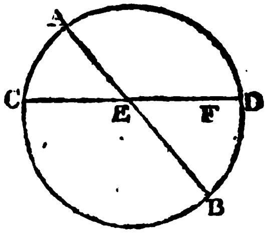
Cor. 2. If ADC be a right angle; the circumference of the circle, of which AC is the diameter, ſhall paſs through the point D. Take any point B in the circumference on the other ſide of AC, and join AB, BC; therefore the angle ABC in the ſemicircle is a right angle [c]; and ABC, ADC are therefore equal to two right angles. Wherefore the circumference ABC paſſes through the point D.

c 31. 3.

PROP. B. THEOR.

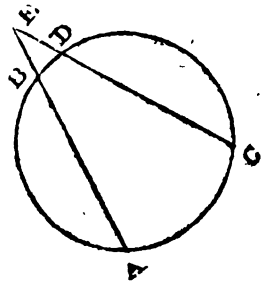
If two ſtraight lines cut one another, and the rectangle contained by the ſegments of one of them be equal to the rectangle contained by the ſegments of the other; the circumference which paſſes through three of their extremities, ſhall alſo paſs through the fourth.

Let the two ſtraight lines AB, CD cut one another in E; and let the rectangle AE, EB be equal to the rectangle CE, ED; alſo let the circumference of a circle paſs through the three points A, C, B; it ſhall alſo paſs through the point D.



If not, let it, if poſſible, cut CD in ſome other point F: and becauſe the ſtraight lines AB, CF in the circle cut one another in E, the rectangle AE, EB is equal [a] to the rectangle CE, EF: but the rectangle AE, EB is equal to the rectangle CE, ED; therefore the rectangle CE, EF is equal to the rectangle CE, ED; and EF is therefore equal to ED; the leſs to the greater, which is impoſſible. Therefore the circumference ACB muſt paſs through the point D. Wherefore, &c. Q. E. D.

a 35. 3.



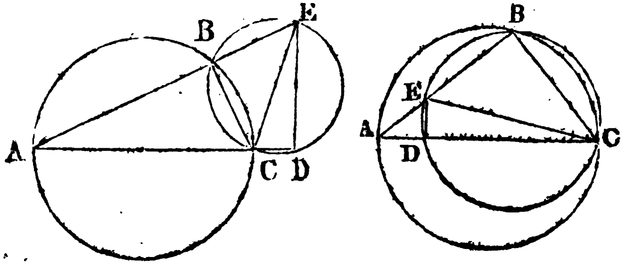
Cor. Likewiſe, if the ſtraight line CD meet a circle in C, and meet a ſtraight line AB, which cuts the circle in the point E without it; and the rectangle CE, ED be equal to the rectangle AE, EB: it may be proved in the ſame manner, that the point D is in the circumference; by uſing the Cor. to the 36th, inſtead of the 35th.

PROP. C. THEOR.

IF a ſtraight line be drawn from the extremity of the diameter of a circle, and meet a perpendicular to that diameter; the rectangle contained by the ſegment of it within the circle, and the ſegment between the extremity of the diameter and the perpendicular, is equal to the rectangle contained by the diameter, and the ſegment of it between the ſame extremity and the perpendicular.

Let ABC be a circle, of which AC is a diameter, and let DE be perpendicular to AC, and from the extremity A, let AB be drawn, meeting the circumference again in B, and the perpendicular in E; the rectangle BA, AE is equal to the rectangle CA, AD.

Join BC, CE; and becauſe ABC is an angle in a ſemicircle,
a 31. 3. it is a right angle [a]; therefore the circle of which CE is a dia-
b 2. Cor. meter, ſhall paſs through B [b]: For the ſame reaſon, it ſhall
A. 3. paſs through the point D: let this be the circle CBED: and



becauſe the ſtraight lines BE, CD, either cut one another in the
point A within the circle, or are drawn from the point A with-
c 35. 3. or out it; the rectangle BA, AE is equal [c] to the rectangle CA,
Cor. 36. 3. AD. Wherefore, &c. Q. E. D.

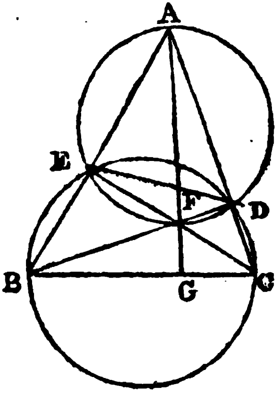
PROP. D. THEOR.

IF from two angles of a triangle, perpendiculars be drawn to the oppoſite ſides, the ſtraight line drawn from the third angle to the point where they meet, ſhall be perpendicular to the third ſide of the triangle.

Let ABC be a triangle, and from the angles at B, C draw BD, CE perpendiculars to AC, AB, and let them meet in F, and join AF meeting BC in G: AC is perpendicular to BC.

Join DE: and becauſe ADF, AEF are right angles, the circumference of the circle, of which AF is the diameter, ſhall paſs through the points E, D [a]; therefore the angle DEF is equal [b] to the angle DAF, becauſe they are in the ſame ſegment DAEF. Again, becauſe BDC, BEC are right angles, the circumference of which BC is the diameter, ſhall paſs through D, E [a]; therefore the angle DEC is equal [b] to the angle DBC, becauſe they are in the ſame ſegment DEBC: but it has been proved, that the angle DEC is equal to the angle DAF; therefore the angle CBF is equal to DAF: and the angle BFG is equal [c] to DFA; therefore the remaining angle BGF is equal [d] to the remaining angle ADF; that is, to a right angle; and AFG is therefore perpendicular to BC. Wherefore, &c. Q. E. D.

a 2. Cor. A. 3.
b 21. 3.
c 15. 1.
d 32. 1.



THE

THE
ELEMENTS
OF
EUCLID.

BOOK IV.

DEFINITIONS.

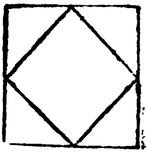
I.

Book IV.

A Rectilineal figure is ſaid to be inſcribed in another rectilineal figure, when all the angles of the inſcribed figure are upon the ſides of the figure in which it is inſcribed, each upon each.

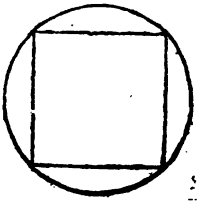
II.

In like manner, a figure is ſaid to be deſcribed about another figure, when all the ſides of the circumſcribed figure paſs through the angular points of the figure about which it is deſcribed, each through each.



III.

A rectilineal figure is ſaid to be inſcribed in a circle, when all the angles of the inſcribed figure are upon the circumference of the circle.

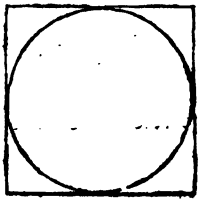


IV.

A rectilineal figure is ſaid to be deſcribed about a circle, when each ſide of the circumſcribed figure touches the circumference of the circle.

V.

In like manner, a circle is ſaid to be inſcribed in a rectilineal figure, when the circumference of the circle touches each ſide of the figure.



VI.

A circle is ſaid to be deſcribed about a rectilineal figure, when the circumference of the circle paſſes through all the angular points of the figure about which it is deſcribed.

VII.

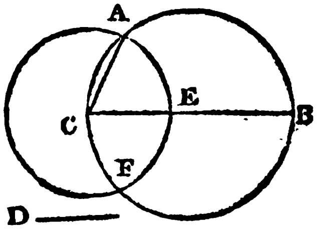
A ſtraight line is ſaid to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

IN a given circle to place a ſtraight line, equal to a given ſtraight line, not greater than the diameter of the circle.

Let ABC be the given circle, and D the given ſtraight line, not greater than the diameter of the circle.

Draw BC, the diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC a ſtraight line BC is placed equal to D: But, if it is not, BC is greater than D; make CE equal [a] to D, and from the centre C, at the diſtance CE, deſcribe the circle AEF, and join CA: Therefore, becauſe C is the centre of the circle AEF, CA is equal to CE; but D is equal to CE; therefore D is equal to CA: Wherefore, in the circle ABC, a ſtraight line is placed equal to the given ſtraight line D, which is not greater than the diameter of the circle. Which was to be done. a 3. 1.



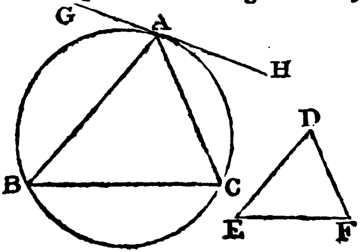
PROP. II. PROB.

IN a given circle, to inſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to inſcribe in the circle ABC a triangle equiangular to the triangle DEF.

Draw [a] the ſtraight line GAH touching the circle in the point A, and at the point A, in the ſtraight line AH, make [b] the angle HAC equal to the angle DEF; and at the point A, in the ſtraight line AG, make the angle GAB equal to the angle DFE, and join BC: Therefore, becauſe HAG touches the circle ABC, and AC is drawn from the point of contact, the a 17. 1. b 23. 1.

the angle HAC is equal [c] to the angle ABC in the alternate ſegment of the circle: But HAC is equal to the angle DEF;
c 32. 3. therefore alſo the angle ABC
is equal to DEF: For the ſame
reaſon, the angle ACB is equal
to the angle DFE; therefore
the remaining angle BAC is
d 32. 1. equal [d] to the remaining angle
EDF: Wherefore the triangle
ABC is equiangular to the triangle DEF, and it is inſcribed
in the circle ABC. Which was to be done.

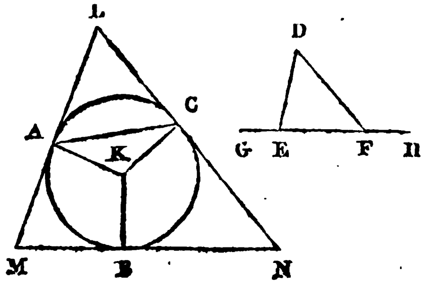


PROP. III. PROB.

See N. ABOUT a given circle, to deſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to deſcribe a triangle about the circle ABC equiangular to the triangle DEF.

Produce EF both ways to the points G, H, and find the
centre K of the circle ABC, and from it draw any ſtraight line
a 23. 1. KB; at the point K, in the ſtraight line KB, make [a] the angle
BKA equal to the angle DEG, and the angle BKC equal to the
angle DFH; and through the points A, B, C, draw the ſtraight
b 17. 3. lines LAM, MBN, NCL, touching [b] the circle ABC: Therefore, becauſe LM, MN, NL touch the circle ABC in the points
A, B, C, to which from the centre are drawn KA, KB, KC,
c 18. 3. the angles at the points A, B, C, are right [c] angles: And becauſe the angles AKB, BKC are equal to DEG, DFH, they
d 16. 1. are greater [d] than DEG, DEF, that is, greater than two right
e 13. 1. angles [e]; therefore, if AC be joined, it ſhall fall between K and
f 2. Cor. L [f]: and the angles ACL, CAL ſhall therefore be leſs than two
15. 1. right angles; wherefore AL, CL ſhall meet [g], if produced: For
g 12.Ax.1. the ſame reaſon, they ſhall, each of them, meet MN: let them
meet in the points L,
M, N: And becauſe the
four angles of the quadrilateral figure AMBK
are equal to four right
angles, for it can be divided into two triangles; and that two of
them KAM, KBM are
right angles, the other
two



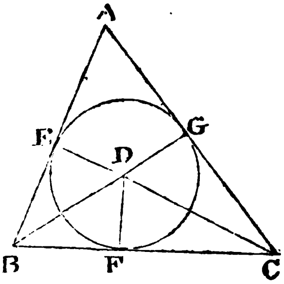
two AKB, AMB are equal to two right angles: But the angles DEG, DEF are likewiſe equal [e] to two right angles; therefore the angles AKB, AMB are equal to the angles DEG, DEF, e 13. 1.
of which AKB is equal to DEG; wherefore the remaining angle AMB is equal to the remaining angle DEF: In like manner, the angle LNM may be demonſtrated to be equal to DFE; and therefore the remaining angle MLN is equal [h] to the remaining angle EDF: h 32. 1.
Wherefore the triangle LMN is equiangular to the triangle DEF: And it is deſcribed about the circle ABC. Which was to be done.

PROP. IV. PROB.

To inſcribe a circle in a given triangle.

Let the given triangle be ABC; it is required to inſcribe a circle in ABC.

Biſect [a] the angles ABC, BCA by the ſtraight lines BD, CD a 9. 1.
meeting one another in the point D, from which draw [b] DE, b 12. 1.
DF, DG perpendiculars to AB, BC, CA: And becauſe the angle EBD is equal to the angle FBD, for the angle ABC is biſected by BD, and that the right angle BED is equal to the right angle BFD, the two triangles EBD, FBD have two angles of the one equal to two angles of the other, and the ſide BD, which is oppoſite to one of the equal angles in each, is common to both; therefore their other ſides ſhall be equal [c]; wherefore c 26. 1.
DE is equal to DF: For the ſame reaſon, DG is equal to DF; therefore the three ſtraight lines DE, DF, DG are equal to one another, and the circle deſcribed from the centre D, at the diſtance of any of them, ſhall paſs through the extremities of the other two; and becauſe the angles at the points E, F, G, are right angles, and the ſtraight line which is drawn from the extremity of a diameter at right angles to it, touches [d] the circle; d 16. 3.
therefore the ſtraight lines AB, BC, CA do each of them touch the circle, and the circle EFG is inſcribed in the triangle ABC. Which was to be done.



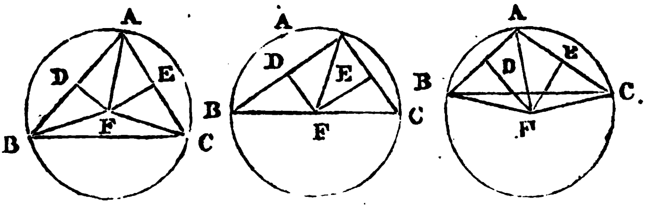
Book IV.

PROP. V. PROB.

See N. TO deſcribe a circle about a given triangle.

Let the given triangle be ABC; it is required to deſcribe a circle about ABC.

a 10. 1. Biſect [a] AB, AC in the points D, E, and from theſe points b 11. 1. draw DF, EF at right angles [b] to AB, AC; DF, EF produced c 12. Ax. 1. meet one another [c]: For, if DE be joined, the angles DEF, EDF are leſs than the two right angles ADF, AEF: Let them meet in F, and join FA; alſo, if the point F be not in BC, join BF, CF: Then, becauſe AD is equal to DB, and DF d 4. 1. common, and at right angles to AB, the baſe AF is equal [d] to the baſe FB: In like manner, it may be ſhown, that CF is equal to FA; and therefore BF is equal to FC; and FA, FB, FC are equal to one another; wherefore the circle deſcribed from the centre F, at the diſtance of one of them, ſhall paſs through the extremities of the other two; and be deſcribed about the triangle ABC. Which was to be done.



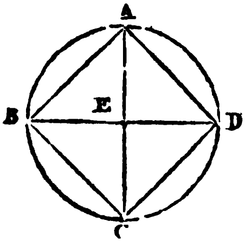
Cor. And it is manifeſt, that, when the centre of the circle falls within the triangle, each of its angles is leſs than a right angle, each of them being in a ſegment greater than a ſemicircle; but, when the centre is in one of the ſides of the triangle, the angle oppoſite to this ſide, being in a ſemicircle, is a right angle; and, if the centre falls without the triangle, the angle oppoſite to the ſide beyond which it is, being in a ſegment leſs than a ſemicircle, is greater than a right angle: Wherefore, if the given triangle be acute angled, the centre of the circle falls within it; if it be a right angled triangle, the centre is in the ſide oppoſite to the right angle; and, if it be an obtuſe angled triangle, the centre falls without the triangle, beyond the ſide oppoſite to the obtuſe angle.

PROP. VI. PROB.

TO inſcribe a ſquare in a given circle.

Let ABCD be the given circle; it is required to inſcribe a ſquare in ABCD

Draw the diameters AC, BD at right angles to one another; and join AB, BC, CD, DA; becauſe BE is equal to ED, for E is the centre, and that EA is common, and at right angles to BD; the baſe BA is equal [a] to the baſe AD; and, for the ſame reaſon, BC, CD are each of them equal to BA or AD; therefore the quadrilateral figure ABCD is equilateral. It is alſo rectangular; for the ſtraight line BD, being the diameter of the circle ABCD, BAD is a ſemicircle; wherefore the angle BAD is a right [b] angle; for the ſame reaſon, each of the angles ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular, and it has been ſhown to be equilateral; therefore it is a ſquare; and it is inſcribed in the circle ABCD. Which was to be done.



a 4. 1.

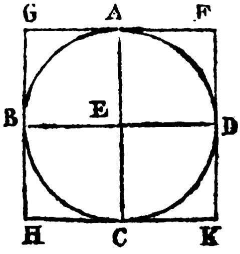
b 31. 3.

PROP. VII. PROB.

TO deſcribe a ſquare about a given circle.

Let ABCD be the given circle; it is required to deſcribe a ſquare about it.

Draw two diameters AC, BD of the circle ABCD, at right angles to one another, and through the points A, B, C, D draw [a] FG, GH, HK, KF touching the circle: and becauſe FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A, the angles at A are right [b] angles; for the ſame reaſon, the angles at the points B, C, D are right angles; and becauſe the angle AEB is a right angle, as likewiſe is EBG, GH is parallel [c] to AC; for the ſame reaſon, AC is parallel to FK; and in like manner, GF, HK may each of them be demonſtrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK are parallelograms; and GF is therefore equal [d] to HK, and GH to FK;



a 17. 3.

b 18. 3.

c 28. 1.

d 34. 1.

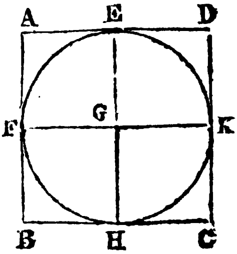
and becauſe AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK: GH FK are each of them equal to GF or HK; therefore the quadrilateral figure FGHK is equilateral. It is alſo rectangular; for GBEA being a parallelogram, and AEB a right angle, d 34. 1. AGB [d] is likewiſe a right angle: In the ſame manner, it may be ſhown, that the angles at H, K, F are right angles; therefore the quadrilateral figure FGHK is rectangular, and it was demonſtrated to be equilateral; therefore it is a ſquare; and it is deſcribed about the circle ABCD. Which was to be done.

PROP. VIII. PROB.

To inſcribe a circle in a given ſquare.

Let ABCD be the given ſquare; it is required to inſcribe a circle in ABCD.

a 10. 1. Biſect [a] each of the ſides AB, AD, in the points F, E, and b 31. 1. through F draw [b] EH parallel to AB or DC, and through F draw FK parallel to AD or BC; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, c 34. 1. and their oppoſite ſides are equal [c]; and becauſe AD is equal to AB, and that AE is the half of AD, and AF the half of AB, AE is equal to AF; wherefore the ſides oppoſite to theſe are equal, viz. FG to GE. In the ſame manner, it may be demonſtrated, that GH, GK are each of them equal to FG or GE; therefore the four ſtraight lines GE, GF, GH, GK, are equal to one another; and the circle deſcribed from the centre G, at the diſtance of one of them, ſhall paſs through the extremities of the other three; and d 29. 1. becauſe the angles at the points E, F, H, K are right [d] angles, and that the ſtraight line which is drawn from the extremity e 16. 3. of a diameter, at right angles to it, touches the circle [e]; therefore each of the ſtraight lines AB, BC, CD, DA touches the circle, which therefore is inſcribed in the ſquare ABCD. Which was to be done.



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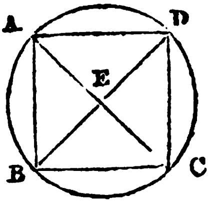
PROP. IX. PROB.

To deſcribe a circle about a given ſquare.

Let ABCD be the given ſquare; it is required to deſcribe a circle about it.

Join AC, BD cutting one another in E; and becauſe DA is equal to AB, and AC common to the triangles DAC, BAC, the two ſides DA, AC are equal to the two BA, AC; and the baſe DC is equal to the baſe BC; wherefore the angle DAC is equal [a] to the angle BAC, and the a 8. 1. angle DAB is biſected by the ſtraight line AC: In the ſame manner, it may be demonſtrated, that the angles ABC, BCD, CDA are ſeverally biſected by the ſtraight lines BD, AC; therefore, becauſe the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and EBA the half of ABC; the angle EAB is equal to the angle EBA; wherefore the ſide EA is equal [b] to the ſide EB: In the ſame manner, it may be de- b 6. 1. monſtrated, that the ſtraight lines EC, ED are each of them equal to EA or EB; therefore the four ſtraight lines EA, EB, EC, ED are equal to one another; and the circle deſcribed from the centre E, at the diſtance of one of them, ſhall paſs through the extremities of the other three, and be deſcribed about the ſquare ABCD. Which was to be done.

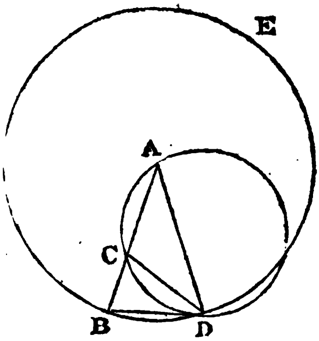


PROP. X. PROB.

To deſcribe an iſoſceles triangle, having each of the angles at the baſe double of the third angle.

Take any ſtraight line AB, and divide [a] it in the point C, ſo a 11. 2. that the rectangle AB, BC be equal to the ſquare of CA; and from the centre A, at the diſtance AB, deſcribe the circle BDE, in which place [b] the ſtraight line BD equal to AC, which is not b 1. 4. greater than the diameter of the circle BDE; join DA, DC, and about the triangle ADC deſcribe [c] the circle ACD; the c 5. 4. triangle ABD is ſuch as is required; that is, each of the angles ABD, ADB is double of the angle BAD.

Becauſe the rectangle AB, BC is equal to the ſquare of AC, and that AC is equal to BD, the rectangle AB, BC is equal to the ſquare of BD; and becauſe from the point B without the circle ACD two ſtraight lines BCA, BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB, BC contained by the whole of the cutting line, and the part of it without the circle is equal to d 37. 3. the ſquare of BD which meets it; the ſtraight line BD touches [d] the circle ACD; and becauſe BD touches the circle, and DC is drawn from the point of contact e 32. 3. D, the angle BDC is equal [e] to the angle DAC in the alternate ſegment of the circle; to each of theſe add the angle CDA; therefore the whole angle BDA is equal to the two angles CDA, DAC; but the exterior angle BCD is f 32. 1. equal [f] to the angles CDA, DAC; therefore alſo BDA is equal to g 5. 1. BCD; but BDA is equal [g] to the angle ABD, becauſe the ſide AD is equal to the ſide AB; therefore ABD, or DBC is equal to BCD; and conſequently the ſide h 6. 1. BD is equal [h] to DC; but BD was made equal to CA; therefore alſo CA is equal to CD, and the angle CDA equal [g] to DAC; therefore the angles CDA, DAC together, are double of the angle DAC: But BCD is equal to the angles CDA, DAC; therefore alſo BCD is double of DAC; and BCD is equal to each of the angles BDA, DBA; each therefore of the angles BDA, DBA is double of the angle DAB; wherefore an iſoſceles triangle ABD is deſcribed, having each of the angles at the baſe double of the third angle. Which was to be done.



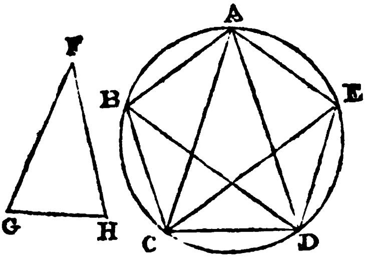
PROP. XI. PROB.

TO inſcribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle; it is required to inſcribe an equilateral and equiangular pentagon in the circle ABCDE.

a 10. 4. Deſcribe [a] an iſoſceles triangle FGH, having each of the angles at G, H, double of the angle at F; and in the circle b 2. 4. ABCDE inſcribe [b] the triangle ACD equiangular to the triangle FGH, ſo that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H; wherefore each of the angles ACD, CDA is double of the c 9. 1. angle CAD: Biſect [c] the angles ACD, CDA by the ſtraight lines CE, DB; and join AB, BC, DE, EA. ABCDE is the pentagon required.

Becauſe each of the angles ACD, CDA is double of CAD, and they are biſected by the ſtraight lines CE, DB, the five angles DAC, ACF, ECD, CDB, BDA are equal to one another: But equal angles ſtand upon equal [d] arches; therefore the five arches AB, d 26. 3. BC, CD, DE, EA are equal to one another: And equal arches are ſubtended by equal [e] ſtraight lines; therefore the five ſtraight lines AB, BC, e 29. 3. CD, DE, EA are equal to one another. Wherefore the pentagon ABCDE is equilateral. It is alſo equiangular; becauſe the arch AB is equal to the arch DE: If to each be added BCD, the whole ABCD is equal to the whole EDCB: And the angle AED ſtands on the arch ABCD, and the angle BAE on the arch EDCB; therefore the angle BAE is equal [f] to the f 27. 3. angle AED: For the ſame reaſon, each of the angles ABC, BCD, CDE is equal to the angle BAE, or AED: Therefore the pentagon ABCDE is equiangular; and it has been ſhown, that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inſcribed. Which was to be done.



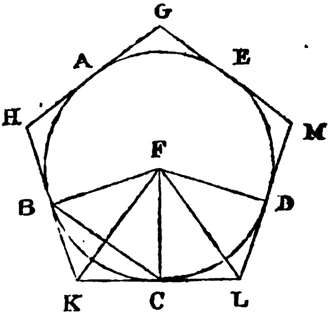
PROP. XII. PROB.

TO deſcribe an equilateral and equiangular pentagon about a given circle. See N.

Let ABCDE be the given circle; it is required to deſcribe an equilateral and equiangular pentagon about the circle ABCDE.

Let the angles of a pentagon, inſcribed in the circle by the laſt propoſition, be in the points A, B, C, D, E, ſo that the arches AB, BC, CD, DE, EA are equal [a]; and through the a 11. 4. points A, B, C, D, E draw GH, HK, KL, LM, MG, touching [b] the circle; take the centre F, and join FB, FK, FC, FL, b 17. 3. FD: And becauſe the ſtraight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F, FC is perpendicular [c] to KL; therefore each of the c 18. 3. angles at C is a right angle: For the ſame reaſon, the angles at the points B, D are right angles: Join BC; and becauſe the angles CBK, BCK are leſs than two right angles, CK, BK

Book IV. do meet [*], if produced: For the ſame reaſon, LM meets KL,
GM, and GH meets GM, HK: And becauſe BF is equal to
*12.Ax.1. FC, and FK common to the triangles BFK, CFK, the two BF,
FK are equal to the two CF, FK; and the baſe BK is equal to
dCor.37.3. the baſe KC, [d] becauſe they are drawn from the point K to
e 8. 1. touch the circle; therefore the angle BFK is equal [e] to the
angle KFC, and the angle BKF to FKC; wherefore the angle
BFC is double of the angle KFC, and BKC double of FKC:
For the ſame reaſon, the angle CFD is double of the angle CFL,
and CLD double of CLF: And becauſe the arch BC is equal
f 27. 3. to CD, the angle BFC is equal [f] to CFD; and BFC is double
of KFC, and CFD double of
CFL; therefore the angle KFC
is equal to CFL; and the right
angle FCK is equal to FCL:
Therefore, in the two triangles
FKC, FLC, there are two angles
of one equal to two of the other,
each to each, and the ſide FC,
adjacent to the equal angles, is
common to both; therefore the
g 26. 1. other ſides ſhall be equal [g] to the
other ſides, and the third angle to
the third angle: Therefore KC is equal to CL, and the angle FKC to FLC: And becauſe KC is equal to CL, KL is double of KC: In the ſame manner, it may be ſhown, that HK is double of BK: And becauſe BK is equal to KC, as was demonſtrated, and that KL is double of KC, and HK double of BK, HK ſhall be equal to KL: In like manner, it may be ſhown, that GH, GM, ML are each of them equal to HK or KL: Therefore the pentagon GHKLM is equilateral. It is alſo equiangular; for, ſince the angle FKC is equal to FLC, and that HKL is double of FKC, and KLM double of FLC, as was before demonſtrated, the angle HKL is equal to KLM: And in like manner, it may be ſhown, that each of the angles KHG, HGM, GML is equal to HKL or KLM: Therefore the pentagon GHKLM is equiangular: And it is equilateral, as was demonſtrated; and it is deſcribed about the circle ABCDE. Which was to be done.

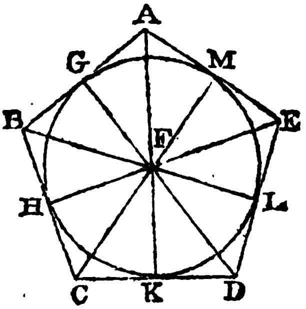


PROP. XIII. PROB.

TO inſcribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inſcribe a circle in it.

Biſect [a] the angles BCD, CDE by the ſtraight lines CF, DF, a 9. 1. and from the point F, in which they meet, draw the ſtraight lines FB, FA, FE: Therefore, ſince BC is equal to CD, and CF common to the triangles BCF, DCF, the two ſides BC, CF are equal to the two DC, CF; and the angle BCF is equal to DCF; therefore the baſe BF is equal [b] to FD, and the angle b 4. 1. CBF to CDF: And becauſe the angle CDE is double of CDF, and that CDE is equal to CBA, and CDF to CBF; CBA is alſo double of CBF; therefore the angle ABF is equal to CBF; wherefore the angle ABC is biſected by BF: In the ſame manner, it may be demonſtrated, that the angles BAE, AED are biſected by AF, FE: From F draw [c] FG, FH, FK, FL, FM perpendiculars to AB, BC, CD, DE, c 12. 1 EA: And becauſe the angle HCF is equal to KCF, and the right angle FHC equal to FKC; in the triangles FHC, FKC there are two angles of one equal to two of the other, and the ſide FC, oppoſite to one of the equal angles in each, is common to both; therefore the other ſides ſhall be equal [d], each to each; wherefore the perpendicular FH is equal d 26. 1. to FK: In the ſame manner, it may be demonſtrated, that FL, FM, FG are each of them equal to FH or FK: Wherefore the circle deſcribed from the centre F, at the diſtance of one of theſe five, ſhall paſs through the extremities of the other four: And becauſe the angles at the points G, H, K, L, M are right angles, and that a ſtraight line from the extremity of a diameter at right angles to it, touches [e] the circle; therefore each of e 16. 3. the ſtraight lines AB, BC, CD, DE, EA touches the circle; wherefore it is inſcribed in the pentagon ABCDE. Which was to be done.



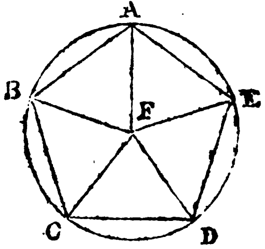
PROP. XIV. PROB.

To deſcribe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to deſcribe a circle about it.

Biſect [a] the angles BCD, CDE by the ſtraight lines CF, FD, a 9. 1. and from the point F, in which they meet, draw the ſtraight

Book IV. lines FB, FA, FE to the points B, A, E. It may be demonſtrated, in the ſame manner as in the preceding propoſition, that the angles CBA, BAE, AED are biſected by the ſtraight lines FB, FA, FE: And becauſe the angle BCD is equal to the angle CDE, and that FCD is the half of the angle BCD, and CDF the half of CDE; the angle FCD is equal to FDC; wherefore the ſide CF is
b 6. 1. equal [b] to the ſide FD: In like manner, it may be demonſtrated, that FB, FA, FE are each of them equal to FC or FD: Therefore the five ſtraight lines FA, FB, FC, FD, FE are equal to one another; and the circle deſcribed from the centre F, at the diſtance of one of them, ſhall paſs through the extremities of the other four, and be deſcribed about the equilateral and equiangular pentagon ABCDE. Which was to be done.



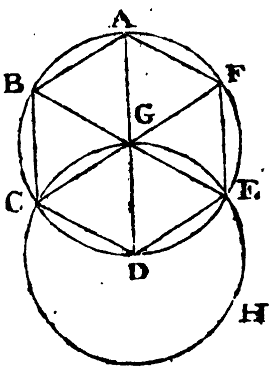
PROP. XV. PROB.

TO inſcribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is required to inſcribe an equilateral and equiangular hexagon in it.

a 1. 3. Find [a] the centre G, and draw the diameter AGD; and in
b 1. 4. the circle place [b] DC, DE, each equal to DG, and join EG, CG, and produce them to the points B, F; and join AB, BC, EF, FA: The hexagon ABCDEF is equilateral and equiangular.

Becauſe DE, EG are each of them equal to DG, the triangle EGD is equilateral; and therefore its three angles EGD, GDE, DEG are equal to one ano-
c Cor. 5. 1. ther [c]; and the three angles of a tri-
d 32. 1. angle are equal [d] to two right angles; therefore the angle EGD is the third part of two right angles: For the ſame reaſon, the angle DGC is alſo the third part of two right angles: And becauſe the adjacent angles EGC,
e 13. 1. CGB are equal [e] to two right angles; the remaining angle CGB is alſo the third part of two right angles; therefore the angles EGD, DGC, CGB are equal to one another: And to theſe



theſe are equal [f] the vertical angles BGA, AGF, FGE: Therefore the ſix angles at the point G are equal to one another: But equal angles ſtand upon equal [g] arches; therefore the ſix arches AB, BC, CD, DE, EF, FA are equal to one another: And equal arches are ſubtended by equal [k] ſtraight lines; therefore the ſix ſtraight lines are equal to one another, and the hexagon ABCDEF is equilateral. It is alſo equiangular; for, ſince the arch AF is equal to ED, to each of theſe add the arch ABCD; therefore the whole arch FABCD ſhall be equal to the whole EDCBA: And the angle FED ſtands upon the arch FABCD, and the angle AFE upon EDCBA; therefore the angle AFE is equal [l] to FED: In the ſame manner, it may be demonſtrated, that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED: Therefore the hexagon is equiangular; and it is equilateral, as was ſhown; and it is inſcribed in the given circle ABCDEF. Which was to be done.

f 15. 1.
g 26. 3.
k 29. 3.
l 27. 3.

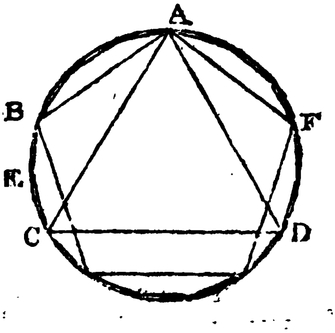
Cor. From this it is manifeſt, that the ſide of the hexagon is equal to the ſtraight line from the centre, that is, to the ſemidiameter or radius of the circle.

And if through the points A, B, C, D, E, F there be drawn ſtraight lines touching the circle, an equilateral and equiangular hexagon ſhall be deſcribed about it, which may be demonſtrated from what has been ſaid of the pentagon; and likewiſe a circle may be inſcribed in a given equilateral and equiangular hexagon, and circumſcribed about it, by a method like to that uſed for the pentagon.

PROP. XVI. PROB.

To inſcribe an equilateral and equiangular quindecagon in a given circle.

Let ABCD be the given circle; it is required to inſcribe an equilateral and equiangular quindecagon in the circle ABCD.



Let AC be the ſide of an equilateral triangle inſcribed [a] in the circle, and AB the ſide of an equilateral and equiangular pentagon inſcribed [b] in the ſame; therefore, of ſuch equal parts as the whole circumference ABCDF contains fifteen, the arch ABC, being

a 2. 4.
b 11. 4.

the third part of the whole, contains five; and the arch AB, which is the fifth part of the whole, contains three; therefore BC, their difference, contains two of the ſame parts: Biſect [c] BC in E; therefore BE, EC are, each of them, the fifteenth part of the whole circumference ABCD: Therefore, if the ſtraight lines BE, EC be drawn, and ſtraight lines equal to them be placed [d] around in the whole circle, an equilateral and equiangular quindecagon ſhall be inſcribed in it. Which was to be done.

c 30. 3.

d 1. 4.

And, in the ſame manner as was done in the pentagon, if, through the points of diviſion, made by inſcribing the quindecagon, ſtraight lines be drawn touching the circle, an equilateral and equiangular quindecagon ſhall be deſcribed about it; And likewiſe, as in the pentagon, a circle may be inſcribed in a given equilateral and equiangular quindecagon, and circumſcribed about it.

THE ELEMENTS OF EUCLID.

BOOK V.

DEFINITIONS.

A.

Book V. See N.

A Magnitude is ſaid to be contained once, in any magnitude not leſs than it, but leſs than its double: and it is ſaid to be contained twice, in any magnitude not leſs than its double, but leſs than its triple; and three times, in any not leſs than its triple, but leſs than its quadruple: and ſo on.

I.

A part of a magnitude, is that which is contained in the magnitude a certain number of times exactly.

II.

A greater magnitude, which contains a leſs a certain number of times exactly, is ſaid to be a multiple of the leſs.

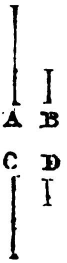
III. Omitted.

B.

Multiplies, which contain their parts the ſame number of times, are called equimultiples of their parts: And the parts are called ſimilar parts of their multiples.

Thus, if A be exactly three times B; then A is ſaid to be a multiple of B: and B is ſaid to be a part of A.

If A be triple of B, and C alſo triple of D; then A, C are called equimultiples of B, D; and B, D are called ſimilar parts of A, C.



IV.

IV.

Magnitudes are ſaid to have a ratio to one another, when the leſs can be multiplied, ſo as to exceed the other; that is, when they are terminated, and of the ſame kind.

C.

In a ratio, the firſt named magnitude is called the Antecedent Term, and the other the Conſequent.

V.

The firſt of four magnitudes is ſaid to have the ſame ratio to the ſecond which the third has to the fourth, when as many times as any multiple of the firſt contains the ſecond, ſo many times does the ſame multiple of the third contain the fourth.

VI.

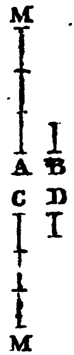
Magnitudes which have the ſame ratio, are called Proportionals.

N. B. When four magnitudes are proportionals, it is uſually expreſſed by ſaying, the firſt is to the ſecond as the third to the fourth.

VII.

If ſome multiple of the firſt contain the ſecond, a greater number of times than the ſame multiple of the third contains the fourth, then the firſt is ſaid to have to the ſecond a greater ratio than the third has to the fourth; and, on the contrary, the third is ſaid to have to the fourth a leſs ratio than the firſt has to the ſecond.

Thus, if of A, C there can be taken ſuch equimultiples MA, MC, that one of them MA contains B oftener than the other MC contains D; then A has to B a greater ratio than C has to D; and C has to D a leſs ratio than A has to B. But if no ſuch equimultiples can be taken; that is, if every multiple of A contains B as many times as the ſame multiple of C contains D, then A is to B as C to D.



VIII. & IX. Omitted.

D.

Magnitudes are ſaid to be continual proportionals, when the firſt has to the ſecond the ſame ratio that the ſecond has to the third; and the ſecond to the third the ſame that the third has to the fourth; and ſo on.

E.

In three proportionals, the ſecond is ſaid to be a mean proportional between the other two; and in any number of proportionals, the firſt and the laſt of them are called the Extremes, and the others are called Means.

X.

The firſt of three proportionals is ſaid to have to the third the duplicate ratio of that which it has to the ſecond.

XI.

Of four continual proportionals, the firſt is ſaid to have to the fourth the triplicate ratio of that which it has to the ſecond; and ſo on, quadruplicate, &c. increaſing the denomination ſtill by unity.

F.

In any number of magnitudes of the ſame kind, the firſt is ſaid to have to the laſt of them the ratio compounded of the ratio of the firſt to the ſecond, and of the ratio of the ſecond to the third, and of the ratio of the third to the fourth, and ſo on to the laſt.

Thus, if A, B, C, D be magnitudes of the ſame kind, the ratio of A to D is ſaid to be compounded of the ratios of A to B, and of B to C, and of C to D, whether theſe ratios be the ſame with one another, or not: but if they be the ſame, the ratio of A to C is alſo ſaid to be duplicate of the ratio of A to B, and the ratio of A to D triplicate of the ratio of A to B.

In like manner, any ratio which is the ſame with that of A to D, is ſaid to be compounded of the ratios of A to B, B to C, and C to D, or of any ratios which are the ſame with them.

XII

In proportionals, one antecedent is ſaid to be homologous to another antecedent, as alſo one conſequent to another.

Changes in the order or magnitude of proportionals are made various ways, ſome of which are the following.

XIII.

By Alternation. When there are four proportionals; it is inferred, by alternation, that the firſt is to the third as the ſecond to the fourth; as is ſhewn in Prop. XVI. Book V.

XIV.

By Inverſion, it is inferred, that the ſecond is to the firſt as the fourth to the third. Prop. B. Book V.

XV.

By Compoſition, it is inferred, that the firſt, together with the ſecond, is to the ſecond, as the third, together with the fourth, is to the fourth. Prop. XVIII. Book V.

XVI.

By Diviſion, it is inferred, that the exceſs of the firſt above the ſecond, is to the ſecond as the exceſs of the third above the fourth is to the fourth. Prop. XVII. Book V.

XVII.

By Converſion, it is inferred, that the firſt is to its exceſs above the ſecond, as the third to its exceſs above the fourth. Prop. E. Book V.

XVIII. & XIX.

By Equality. When there are two ranks, each of them containing the ſame number of magnitudes more than two, and theſe magnitudes are proportionals, when taken two and two in a direct order in each rank; that is, the firſt to the ſecond of the firſt rank, as the firſt to the ſecond of the other rank; and the ſecond to the third, as the ſecond to the third; and ſo on; then it is inferred, by equality, that the firſt is to the laſt of the firſt rank as the firſt of the other rank to the laſt. Prop. XXII. Book V.

XX.

By Perturbate Equality. When there are two ranks as before, and the magnitudes are proportionals when taken two and two in each rank, one in a direct, and the other in an inverſe order; that is, the firſt to the ſecond of the firſt rank, as the laſt but one to the laſt of the other rank; and the ſecond to the third, as the laſt but two to the laſt but one; and ſo on; then it is inferred, by perturbate equality, that the firſt is to the laſt of the firſt rank, as the firſt to the laſt of the other rank. Prop. XXIII. Book V.

Thus, if A, B, C, D, be magnitudes in one rank, and E, F, G, H, as many in another; then, if A be to B as E to F, and B to C as F to G, and C to D as G to H; it is inferred, by equality, that A is to D as E to H. But if A be to B as G to H, and B to C as F to G, and C to D as E to F, it is inferred, by perturbate equality, that A is to D as E to H.

A, B, C, D.
E, F, G, H.

AXIOMS.

I.

Equal magnitudes contain the ſame magnitude, the ſame number of times; and the ſame contains equals the ſame number of times.

II.

That magnitude which contains the ſame a greater number of times than another does, is greater than that other.

III.

That magnitude which is contained a greater number of times than another, in the ſame magnitude, is leſs than that other.

IV.

Equimultiples of the ſame, or of equal magnitudes, are equal to one another.

V.

A part of a greater magnitude is greater than the ſame part of a leſs.

PROP. I. THEOR.

IF any number of magnitudes contain as many others the ſame number of times; all the firſt magnitudes taken together ſhall contain all the others that ſame number of times: and if each of the firſt contain its other exactly, the whole ſhall contain the whole exactly.

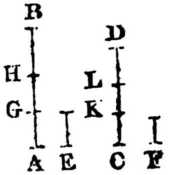
Let any number of magnitudes AB, CD contain as many E, F the ſame number of times; as many times as AB contains E, ſo many times ſhall AB and CD together contain E and F together.

If AB contain E but once, it is leſs [a] than double of E: and CD is leſs than double of F; therefore AB, CD together are leſs than double of E, F together; and therefore they contain E, F but once.

a Def. A. 5.

If AB contain E more than once, from it cut off AG, GH equal each of them to E, and from CD cut off CK, KL equal to F, ſo that the remainders BH, DL contain E, F but once; therefore AB contains E as many times as there are magnitudes AG, GH, HB; and CD contains F as many times as there are magnitudes CK, KL, LD: but AB, CD contain E, F equally; therefore the number of the magnitudes AG, GH, HB is equal to the number of the others CK, KL, LD: and becauſe HB contains E once, and LD contains F once; therefore BH and DL together contain E and F once: and becauſe GH is equal to E, and KL to F; therefore GH, and KL together are equal [b] to E, and F together. For the ſame reaſon, AG and CK together are equal to E and F together. Wherefore as many times as E is contained in AB, ſo many times are E, F together contained in AB, CD together.

b 2. Ax. 1.



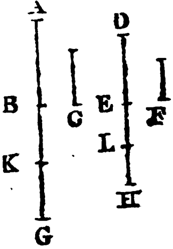
Likewiſe, if BH be equal to E, and DL to F; BH, DL together are equal to E, F together: and GH, KL are equal to E, F; as alſo are AG, CK; therefore AB, CD together contain E, F exactly; that is, AB together with CD is the ſame multiple of E together with F, that AB is of E. The ſame demonſtration holds if there be more magnitudes. Wherefore, &c. Q. E. D.

PROP. II. THEOR.

IF to two magnitudes which contain two others the ſame number of times, equimultiples of theſe others be added; the wholes ſhall contain theſe others the ſame number of times: and if the firſt two be equimultiples of the others, the wholes ſhall alſo be equimultiples of them.

Let AB contain C the ſame number of times that DE contains F; and let BG be the ſame multiple of C that EH is of F: Then ſhall the whole AG contain C the ſame number of times that the whole DH contains F.

Becauſe BG is the ſame multiple of C, that EH is of F, as a 2.Def.5. many magnitudes as are in BG equal to C, ſo many [a] are there in EH equal to F. Divide BG into BK, KG, each of them equal to C, and EH into EL, HL, each equal to F: therefore the number of the magnitudes BK, KG is equal to the number of the others EL, LF: and becauſe AB contains C the ſame number of times that DE contains F, and that BK is equal to C, and EL to F; therefore AK contains C the ſame number of times that DL contains F: and KG is equal to C, and LH to F; therefore AG contains C the ſame number of times that DH contains F. Likewiſe, if AB contain C exactly, AK and AG contain C exactly; and if DE contain F exactly, DL and DH contain F exactly; therefore, if AB, DE be equimultiples of C, F; AG, DH are alſo equimultiples of them. Wherefore, &c. Q. E. D.



PROP. III. THEOR.

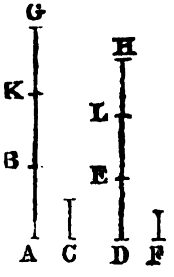
IF two magnitudes be equimultiples of two others; any equimultiples of the firſt two are alſo equimultiples of the other two.

Let AB be the ſame multiple of C that DE is of F, and let AG, DH be equimultiples of AB, DE; then AG is the ſame multiple of C that DH is of F.

a 2.Def.5. Let the magnitudes equal [a] to AB, in AG, be AB, BK, KG; and thoſe equal to DE, in DH, be DE, EL, LH: and becauſe AG, DH are equimultiples of AB, DE, the number of the b Def.B.5. magnitudes AB, BK, KG is equal [b] to the number of the others

DE, EL, LH: And becauſe AB is the ſame multiple of C that DE is of F, and that BK is equal to AB, and EL to ED; therefore BK is the ſame multiple of C that EL is of F: For the ſame reaſon, KG is the ſame multiple of C that LH is of F: And becauſe to AB, DE, which are equimultiples of C, F, there are added BK, EL, which are alſo equimultiples of C, F; therefore the whole AK is the ſame multiple [c] of C that the whole DL is of F. For the ſame reaſon, AG is the ſame multiple of C that DH is of F. Wherefore, &c. Q. E. D.

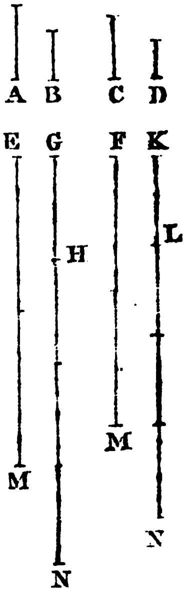
c 2. 5.



PROP. IV. THEOR.

IF the firſt of four magnitudes has the ſame ratio to the ſecond, which the third has to the fourth, and any equimultiples be taken of the firſt and third, and alſo any equimultiples of the ſecond and fourth: the multiple of the firſt ſhall have to that of the ſecond, the ſame ratio which the multiple of the third has to that of the fourth.

Let A the firſt, have to B the ſecond, the ſame ratio which the third C has to the fourth D; and of A and C, let E and F be any equimultiples; and of B and D, let GH and KL be any equimultiples: then ſhall E have the ſame ratio to GH that F has to KL.



Take ME any multiple of E greater than GH, and MF the ſame multiple of F: alſo take NG, the leaſt multiple of GH, that is greater than ME, and NK the ſame multiple of KL: Therefore NH, NL are either equal to GH, KL, or equimultiples of them: But GH, KL are themſelves equimultiples of B, D; therefore NH, NL are equimultiples [a] of B, D: For the ſame reaſon, NG, NK are equimultiples of B, D; and ME, MF, of A, C: and becauſe, as A to B, ſo is C to D, and ME, MF are equimultiples of A, C; ME contains B the ſame number of times [b] that MF contains D: NG alſo and NK contain B and D the ſame number of times [c], becauſe they are equimultiples of them; but ME does not contain B ſo many times as NG contains

a 3. 5.
b 5. Def. 5.
c A. Def. 5.

contains B, for ME is leſs than NG; therefore MF does not contain D ſo many times as NK does: MF is therefore leſs [d] d 2. Ax. 5. than KN. In like manner, becauſe ME is not leſs than NH, for NG is the leaſt multiple of GH greater than ME, it may be proved, that MF is not leſs than NL: and becauſe ME is not leſs than NH, but leſs than NG the next greater multiple of e def. A. 5. GH; ME and NH contain GH the ſame number of times [e]. For the ſame reaſon, MF and NL contain KL the ſame number of times: but NH, NL contain GH, KL equally, for they are either equal to GH, KL, or equimultiples of them; therefore ME, MF contain GH, KL equally: and ME, MF are any equimultiples whatever of E, F; as many times therefore as any multiple of E contains GH, ſo many times does the ſame b 5. Def. 5. multiple of F contain KL: Wherefore, as E is to GH, ſo is F [b] to KL. Therefore, if the firſt, &c. Q. E. D.

Cor. Likewiſe, if the firſt be to the ſecond, as the third to the fourth, then alſo any multiple of the firſt is to the ſecond, as the ſame multiple of the third to the fourth: and in like manner, the firſt is to any multiple of the ſecond, as the third to the ſame multiple of the fourth.

For, it was proved in the propoſition, that as many times as ME any multiple of E, contains B, ſo many times does MF the ſame multiple of F, contain B; therefore, as E is to B, ſo is [b] F to D.

And if ME, MF be any equimultiples of A, C, of which ME is greater than GH; and NG, NK be taken as in the propoſition: it may be demonſtrated, as before, that ME contains GH the ſame number of times that MF contains KL: therefore, as A to GH, ſo is [b] C to KL.

PROP. V. THEOR.

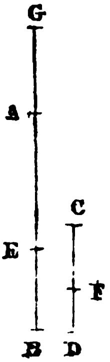
IF one magnitude be the ſame multiple of another, that a magnitude taken from the firſt is of one taken from the other; the remainder ſhall be the ſame multiple of the remainder, that the whole is of the whole.

Let the magnitude AB be the ſame multiple of CD, that AE taken from the firſt is of CF taken from the other; the remainder EB ſhall be the ſame multiple of the remainder FD, that the whole AB is of the whole CD.

Take AG the ſame multiple of FD, that AE is of CF: a 1. 5. therefore AE is the ſame multiple [a] of CF, that EG is of CD:

But AE, by the hypotheſis, is the ſame multiple of CF that AB is of CD: Therefore EG is the ſame multiple of CD that AB is of CD; wherefore EG is equal [b] to AB: Take from them the common magnitude AE, and the remainder AG is equal to the remainder EB. Wherefore, ſince AE is the ſame multiple of CF that AG is of FD, and that AG is equal to EB; therefore AE is the ſame multiple of CF that EB is of FD: But AE is the ſame multiple of CF, that AB is of CD; therefore EB is the ſame multiple of FD that AB is of CD. Therefore, &c. Q. E. D.

b 4. Ax. 5.



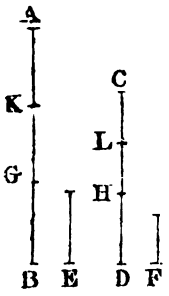
PROP. VI. THEOR.

IF from two magnitudes, which contain two others the ſame number of times, there be taken equimultiples of theſe others; the remainders, if they be not leſs than the other magnitudes, ſhall contain them the ſame number of times.

Let the magnitude AB contain E the ſame number of times that CD contains F; and let AG, CH, equimultiples of E, F, be taken from AB, CD, ſo that the remainder GB be not leſs than E: GB, HD ſhall contain E, F the ſame number of times.

Becauſe AG is the ſame multiple of E that CH is of F, there are as many magnitudes in AG equal to E, as there are in CH equal to F [a]: Divide AG into AK, KG, each equal to E, and CH into CL, LH, each equal to F; therefore the number of the firſt AK, KG is equal to the number of the laſt CL, LH: and becauſe AB contains E the ſame number of times that CD contains F; and AK, CL are equal to E, F; the remainder BK contains E the ſame number of times that the remainder DL contains F. In like manner, if from BK, DL, which contain E, F equally, there be taken KG, LH, which are equal to E, F; the remainder BG contains E the ſame number of times that the remainder DH contains F. Wherefore, &c. Q. E. D.

a Def. B. 5.

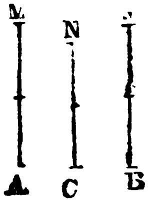


PROP.

PROP. VII. THEOR.

EQUAL magnitudes have the ſame ratio to the ſame magnitude; and the ſame has the ſame ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other; A has the ſame ratio to C that B has to C; and C has the ſame ratio to A that it has to B.



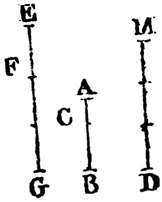
Take MA, MB any equimultiples of A, B: and becauſe A is equal to B, and MA, MB are equimultiples of them, MA is
a 4. Ax. 5. equal [a] to MB: and equal magnitudes contain
b 1. Ax. 5. the ſame equally [b]; therefore MA, MB contain C the ſame number of times: and MA, MC are any equimultiples of A, B; as many times therefore as any multiple of A contains C, ſo many times does the ſame multiple of B contain C.
c 5. Def. 5. Therefore, as A is to C, ſo is [c] B to C.

Likewiſe, C has the ſame ratio to A that it has to B. Take NC any multiple of C greater than A: and becauſe A is equal to B, NC contains each of them the ſame number of times [b]; that is, any multiple of C contains A the ſame number of times that it contains B; therefore C is to A, as C is [c] to B. Wherefore, &c. Q. E. D.

PROP. VIII, THEOR.

OF unequal magnitudes, the greater has a greater ratio to the ſame than the leſs has; and the ſame has a greater ratio to the leſs, than it has to the greater.

Let AB, BC be unequal magnitudes, of which AB is the greater, and let D be any magnitude whatever: AB has a greater ratio to D than BC to D: and D has a greater ratio to BC than to AB.



Take EF, FG ſuch equimultiples of AC, CB, that each of them may be greater than D: and becauſe EF is the ſame multiple of AC that FG is of CB;
a 1. 5. the whole EG is the ſame multiple [a] of the whole AB that FG is of CB: but, becauſe EF is greater than D, EG contains D a greater number of times than FG contains it. Wherefore a multiple of AB is found, viz. EG, which contains D a greater number of times than FG,

the ſame multiple of BC contains D; therefore AB has to D a greater ratio [b] than BC has to D.

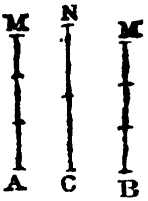
Alſo D has to BC a greater ratio than it has to AB. For, having made the ſame conſtruction, take alſo MD the leaſt multiple of D, that is not leſs than FG: therefore MD is either equal to FG, or exceeds it by a magnitude leſs than D: but EF is greater than D; therefore MD is leſs than EG: but EG, FG contain AB, BC the ſame number of times exactly; therefore MD does not contain AB ſo many times as it contains BC; that is, a multiple of D is taken, which contains BC a greater number of times than it contains BA: therefore D has to BC a greater ratio [b] than it has to AB. Wherefore, &c. Q. E. D.

[b] 7. def. 5.

PROP. IX. THEOR.

MAgnitudes which have the ſame ratio to the ſame magnitude, are equal to one another; and thoſe to which the ſame magnitude has the ſame ratio, are equal to one another.

Let A have the ſame ratio to C that B has to C; A is equal to B: For if not, let A be the greater; then, by what was ſhown in the preceding propoſition, there is ſome multiple of A which contains C a greater number of times than the ſame multiple of B does. Let MA be that multiple of A which contains C a greater number of times than MB, the ſame multiple of B contains C. But, becauſe A is to C as B is to C, and MA, MB are equimultiples of A, B; MA, MB contain C the ſame number of times [a]; but MA contains C a greater number of times than MB does; which is impoſſible; A therefore and B are not unequal; that is, they are equal.



[a] 5. Def. 5.

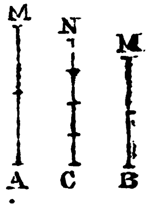
Next, let C have the ſame ratio to A that it has to B; A is equal to B. For, if not, let A be the greater; therefore, as was ſhewn in Prop. VIII. there is ſome multiple NC of C, which contains B a greater number of times than it contains A: but becauſe C is to B, as C is to A, and that NC is a multiple of C; NC contains B and A the ſame number of times [a]: but it ntains B oftener; which is impoſſible. Therefore A is equal B. Wherefore, &c. Q. E. D.

PROP.

PROP. X. THEOR.

THAT magnitude which has a greater ratio than another has to the ſame magnitude, is the greater of the two: and that magnitude to which the ſame has a greater ratio than it has to another, is the leſſer of the two.

Let A have to C a greater ratio than B has to C; A is greater than B: For, becauſe A has a greater ratio to C, than B has to C, there is ſome multiple of A which contains C a greater number of times than the ſame multiple of B [a]: Let MA be that multiple of A which contains C a greater number of times than MB, the ſame multiple of B contains C; therefore MA is greater [b] than MB: and A, B are the ſame parts of them; therefore A is greater [c] than B.



a 7. Def. 5.
b 2. Ax. 5.
c 5. Ax. 5.

Next, let C have a greater ratio to B than it has to A; B is leſs than A: For, there is ſome multiple NC of C, which contains B a greater number of times than it contains A; therefore B is leſs [d] than A. Wherefore, &c. Q. E. D.

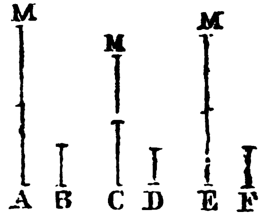
d 3. Ax. 5.

PROP. XI. THEOR.

RATIOS that are equal to the ſame ratio, are equal to one another.

Let A be to B, as C is to D, and as C to D, ſo let E be to F; A is to B, as E to F.

Take MA, MC, ME any equimultiples of A, C, E: and becauſe A is to B as C to D, and MA, MC are equimultiples of A, C; MA contains B the ſame number of times [a] that MC contains D: and becauſe C is to D as E is to F, and MC, ME are equimultiples of C, E; MC contains D the ſame number of times, [a] that ME contains F: But MC contains D the ſame number of times that MA contains B; therefore MA contains B the ſame number of times that ME contains F: and MA, ME are any equimultiples of A, E; as many times, therefore, as any multiple of A contains B, ſo many times does the ſame multiple of E contain F: therefore, as A is to B, ſo [a] is E to F. Wherefore, &c. Q. E. D.



a 5. Def. 5.

PROP.

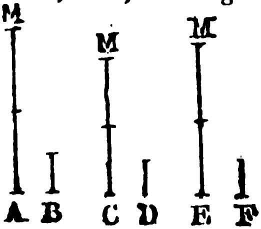
PROP. XII. THEOR.

IF any number of magnitudes be proportionals, as one of the antecedents is to its conſequent, ſo ſhall all the antecedents taken together be to all the conſequents.

Let any number of magnitudes A, B, C, D, E, F, be proportionals; that is, as A is to B, ſo C to D, and E to F; as A is to B, ſo ſhall A, C, E together be to B, D, F together.

Take MA, MC, ME any equimultiples of A, C, E: therefore MA is the ſame multiple of A, that MA, MC, ME together is of A, C, E together [a]: And becauſe A is to B, as C is to D, and as E to F; and that MA, MC, ME are equimultiples of A, C, E; MA contains B as many times [b] as MC contains D, and ME contains F: Wherefore, as many times as MA contains B, ſo many times ſhall MA, MC, ME together contain B, D, F together [a]: But MA, and MA, MC, ME together, are any equimultiples of A, and of A, C, E together; as many times, therefore, as any multiple of A contains B, ſo many times does the ſame multiple of A, C, E together contain B, D, F together: therefore, as A is to B, ſo [b] are A, C, E together to B, D, F together. Wherefore, &c. Q. E. D.

a 1. 5.
b 5. Def. 5.



PROP. XIII. THEOR.

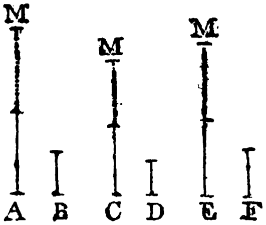
IF the firſt has to the ſecond the ſame ratio that the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the ſixth; the firſt ſhall alſo have to the ſecond a greater ratio than the fifth has to the ſixth.

Let A the firſt have the ſame ratio to B the ſecond, which C the third has to D the fourth, but C the third, to D the fourth, a greater ratio than E the fifth, to F the ſixth: Alſo the firſt A ſhall have to the ſecond B a greater ratio than the fifth E to the ſixth F.

Becauſe C has a greater ratio to D, than E to F, there is ſome multiple of C which contains D, a greater number of times [a] than the ſame multiple of E contains F: Let MC be that multiple of C, which contains D a greater number of times than

a 7. Def. 5.

ME, the ſame multiple of E, contains F; and whatever multiple
MC is of C, take MA the ſame multiple of A: then, becauſe
A is to B, as C to D, and MA, MC
are equimultiples of A, C; MA con-
b 5. Def. 5. tains B the ſame nnmber of times[b]
that MC contains D: but MC con-
tains D a greater number of times
than ME contains F; therefore MA
contains B a greater number of times
than ME contains F; and MA, ME
are equimultiples of A, E; there is
therefore ſome multiple of A which contains B a greater num-
ber of times than the ſame multiple of E contains F: there-
a 7. Def. 5. fore A has a greater ratio to B[a], than E has to F. Where-
fore, &c. Q. E. D.



Cor. And if the firſt has a greater ratio to the ſecond, than the third has to the fourth, but the third the ſame ratio to the fourth, which the fifth has to the ſixth; it may be demonſtrated in like manner, that the firſt has a greater ratio to the ſecond, than the fifth has to the ſixth.

PROP. XIV. THEOR.

If the firſt has to the ſecond the ſame ratio which the third has to the fourth; then, if the firſt be greater than the third, the ſecond ſhall be greater than the fourth; and if equal, equal; and if leſs, leſs.

Let the firſt A have to the ſecond B, the ſame ratio which the third C has to the fourth D; if A be greater than C, B is greater than D.

Becauſe A is greater than C, and B is any other magnitude,
a 8. 5. A has to B a greater ratio than C to B[a]:
But, as A is to B, ſo is C to D; therefore
alſo C has to D a greater ratio than C has to
b 13. 5. B[b]: But of two magnitudes, that to which
c 10. 5. the ſame has the greater ratio is the leſſer[c]:
Wherefore D is leſs than B; that is, B is
greater than D.

Secondly, If A be equal to C, B is equal to D: For A is to
d 9. 5. B, as C, that is A, is to D; therefore B is equal to D[d].

Thirdly, If A be leſs than C, B ſhall be leſs than D: For C is greater than A; and becauſe C is to D, as A to B, and C greater than A, D is greater than B, by the firſt caſe; therefore B is leſs than D. Wherefore, &c. Q. E. D.

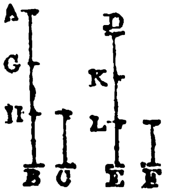
PROP.

PROP. XV. THEOR.

MAgnitudes have the ſame ratio to one another which their equimultiples have.

Let AB be the ſame multiple of C, that DE is of F: C is to F, as AB to DE.

Becauſe AB is the ſame multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F: Let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB; and DE into magnitudes, each equal to F, viz. DK, KL, LE: then the number of the firſt AG, GH, HB, ſhall be equal to the number of the laſt DK, KL, LE: and becauſe AG, GH, HB are all equal, and that DK, KL, LE are alſo equal to one another; therefore AG is to DK, as GH to KL, and as HB to LE [a]; a 7. 5. and as one of the antecedents to its conſequent, ſo are all the antecedents together to all the conſequents together [b]; wherefore, as AG is to DK, ſo is AB to DE: b 12. 5. But AG is equal to C, and DK to F; therefore, as C is to F, ſo is AB to DE. Therefore, &c. Q. E. D.

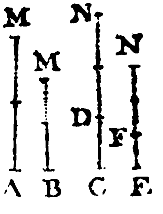


PROP. XVI. THEOR.

IF four magnitudes of the ſame kind be proportionals, they ſhall alſo be proportionals when taken alternately.

Let the four magnitudes A, B, CD, EF be proportionals, viz. as A to B, ſo CD to EF: They ſhall alſo be proportionals when taken alternately; that is, A ſhall be to CD, as B to EF.

Take MA, MB any equimultiples of A, B, and take NC the leaſt multiple of CD that is greater than MA, and NE the ſame multiple of EF: and becauſe MA, MB are equimultiples of A, B, and that magnitudes have the ſame ratio to one another, which their equimultiples have [a]; a 15. 5. therefore A is to B, as MA to MB: But as A is to B, ſo is CD to EF; wherefore CD is to EF, as [b] MA to MB. b 11. 5. Again, becauſe NC, NE are equimultiples of CD, EF, as CD to EF, ſo is NC to NE [a]: but as CD to EF, ſo is MA to MB; therefore MA is to MB, as NC to NE [b]: and the firſt MA is leſs than the third NC; therefore the ſecond



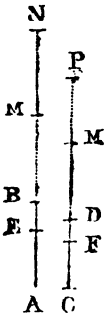
MB is leſs [c] than the fourth NE. In like manner, becauſe MA is not leſs than ND, it may be proved, that MB is not leſs than c 14. 5 NF: Therefore MA, MB contain CD, EF the ſame number of d A.Def.5. times that ND, NF contain them [d]: But ND, NF contain CD, EF equally, for they are either equal to CD, EF, or equimultiples of them; therefore MA, MB contain CD, EF equally: and MA, MB are any equimultiples of A, B: as many times, therefore, as any multiple of A contains CD, ſo many times does the ſame multiple of B contain EF: therefore A is to CD, e 5. Def. 5. as B to EF [e]. Wherefore, &c. Q. E. D.

PROP. XVII. THEOR.

IF magnitudes, taken jointly, be proportionals, they ſhall alſo be proportionals when taken ſeparately; or by diviſion.

Let AB, BE, CD, DF be the magnitudes taken jointly, which are proportionals; that is, as AB to BE, ſo is CD to DF; they ſhall alſo be proportionals taken ſeparately, viz. as AE to EB, ſo CF to FD.

Take of AE, EB, CF, FD any equimultiples MA, MN, MC, MP, ſo that MA be greater than BE: and becauſe MA is the ſame multiple of AE, that MN is of EB, the whole NA a 1. 5. is the ſame multiple of AB, that MA is of AE [a]: but MA is the ſame multiple of AE, that MC is of CF; wherefore NA is the ſame multiple of AB, that MC is of CF. Again, becauſe MC is the ſame multiple of CF, that MP is of FD; therefore PC is the ſame multiple of CD, that MC is of CF [a]: but NA, MC are equimultiples of AB, CF; therefore NA, PC are equimultiples of AB, CD: and becauſe AB is to BE, as CD to DF, and that NA, PC are equimultiples of AB, CD, b 5. Def. 5. NA contains BE the ſame number of times [b] that PC contains DF: And from NA, PC are taken NM, PM, which are equimultiples of BE, DF; therefore the c 6. 5. remainder MA contains BE the ſame number of times [c] that the remainder MC contains DF: and MA, MC are any equimultiples of AE, CF; as many times, therefore, as any multiple of AE contains EB, ſo many does the ſame multiple of CF contain FD; therefore, as AE is to EB, ſo [b] is CF to FD. Wherefore, &c. Q. E. D.

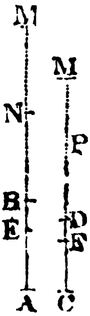


PROP. XVIII. THEOR.

IF magnitudes, taken ſeparately, be proportionals, they ſhall alſo be proportionals when taken jointly; that is, by compoſition.

Let AE be to EB, as CF to FD; they ſhall alſo be proportionals when taken jointly; that is, as AB to BE, ſo CD to DF.

Take of AB, BE, CD, DF any equimultiples MA, MN, MC, MP: And becauſe the whole MA is the ſame multiple of the whole AB, that MN is of BE; the remainder NA is the ſame multiple of the remainder AE, that the whole MA is of the whole AB [a]. For the ſame reaſon, PC is the ſame multiple a 5. 5. of CF, that MC is of CD: But MA, MC are equimultiples of AB, CD; therefore NA, PC are equimultiples of AE, CF: and becauſe AE is to EB, as CF to FD, and that NA, PC are equimultiples of AE, CF: therefore NA contains BE the ſame number of times that PC contains DF [b]: and MN, MP b 5. Def. 5. are equimultiples of the ſame BE, DF; therefore the whole MA contains BE the ſame number of times that the whole MC contains DF [c]: and MA, MC c 2. 5. are any equimultiples of AB, CD; as many times, therefore, as any multiple of AB contains BE, ſo many times does the ſame multiple of CD contain DF: Wherefore, as AB is to BE, ſo [b] is CD to DF. Wherefore, &c. Q. E. D.

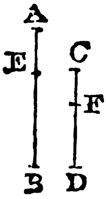


PROP. XIX. THEOR.

IF a whole magnitude be to a whole, as a magnitude taken from the firſt is to a magnitude taken from the other; the remainder ſhall be to the remainder, as the whole to the whole.

Let the whole AB, be to the whole CD, as AE taken from AB, to CF taken from CD; the remainder EB ſhall be to the remainder FD, as the whole AB to the whole CD.

Becauſe AB is to CD, as AE to CF; likewiſe, alternately [a], BA is to AE, as DC to CF: and be- a 16. 5. cauſe, if magnitudes, taken jointly, be proportionals, they are alſo proportionals [b] when taken ſepa- b 17. 5. rately; therefore, as BE is to EA, ſo is DF to FC; and alternately [a], as BE is to DF, ſo is EA to FC: But, as EA to FC, ſo, by the hypotheſis, is AB to CD;



CD; therefore alſo BE, the remainder, ſhall be to the remainder DF, as the whole AB to the whole CD: Wherefore, &c. Q. E. D.

Cor. If the whole be to the whole, as a magnitude taken from the firſt is to a magnitude taken from the other; the remainder likewiſe is to the remainder, as the magnitude taken from the firſt to that taken from the other: The demonſtration is contained in the preceding.

PROP. A. THEOR.

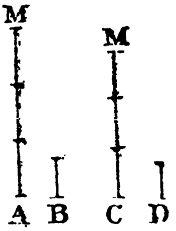
IF the firſt of four magnitudes has to the ſecond, the ſame ratio which the third has to the fourth; then, if the firſt be greater than the ſecond, the third is alſo greater than the fourth; and if equal, equal; and if leſs, leſs.

Let A be to B, as C is to D; if A be greater than B, C is greater than D.

Becauſe A is greater than B, take, as in the eighth propoſition, MA a multiple of A, that ſhall contain B a greater number of times than it contains A, and take MC the ſame multiple of C: and becauſe A is to B, as C is to D; MC contains D the ſame number of

a 5. Def. 5. times [a] that MA contains B: and this number is greater than the number of times that MA contains A, or MC contains C; therefore D is contained a greater number of times than C is in the ſame magnitude MC: wherefore D is

b 3. Ax. 5. leſs than C [b]; that is, C is greater than D.



Next, If A be leſs than B, C is leſs than D: Take MA, MC any equimultiples of A, C: and becauſe A is leſs than B, MA contains A a greater number of times than it contains B; therefore, as before, MC contains C a greater number of times than it contains D; C is therefore leſs than D [b].

Laſtly, If A be equal to B, C is equal to D: For, becauſe C is to D, as A is to B, if C were greater than D, or leſs than it, A would be greater than B, or leſs than it, by the former caſes; but it is not: Therefore C is equal to D. Wherefore, &c. Q. E. D.

Otherwiſe,

Take E equal to D: and let A be greater than B; then, as in the eighth propoſition, ſome multiple of A can be found, which contains B a greater number of times than the ſame multiple of E contains D;

A, B, C, D,
E.

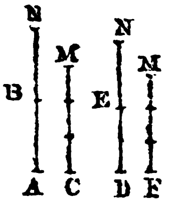
D[c]; therefore A has to B a greater ratio than E has to D[c]:
but C is to D, as A to B; therefore C has to D a greater ratio
than E has to D[d]; wherefore C is greater[e] than E or D. c 7. Def. 5.
In like manner, it may be proved, that if A be equal to B, d 13. 5.
C is equal to D; and if leſs, leſs. e 10. 5.

PROP. B. THEOR.

IF four magnitudes be proportionals, they are alſo proportionals when taken inverſely.

Let AB be to C, as DE is to F; then alſo, inverſely, C is to AB, as F to DE.

Take MC, MF any equimultiples of C, F; and take NA the
leaſt multiple of AB that is greater than MC, and ND the ſame
multiple of DE: and becauſe AB is to C, as
DE to F, and NA, ND are equimultiples of
AB, DE, and MC, MF equimultiples of C,
F; therefore NA is to MC, as ND to MF[a]: a 4. 5.
and the firſt NA is greater than the ſecond
MC, therefore the third ND is greater than
the fourth MF[b]; that is, MF is leſs than ND. b A. 5.
In like manner, becauſe MC is not leſs than
NB, it may be proved, that MF is not leſs than NE: Wherefore
MC, MF contain AB, ED the ſame number of times that
NB, NE contain them: But NB, NE contain AB, DE equally,
for they are either equal to AB, DE, or equimultiples of
them; therefore MC, MF contain AB, DE equally: and MC,
MF are any equimultiples of C, F; as many times therefore, as
any multiple of C contains AB, ſo many times does the ſame
multiple of F contain DE; therefore, as C is to AB, ſo[c] is F to c 5. Def. 5.
DE. Wherefore, &c. Q. E. D.



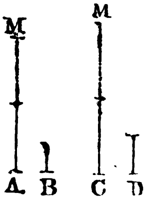
PROP. C. THEOR.

IF the firſt be the ſame multiple of the ſecond, or the ſame part of it, that the third is of the fourth; the firſt is to the ſecond, as the third is to the fourth.

Let the firſt A be the ſame multiple of B the ſecond, that C the third is of the fourth D: A is to B, as C is to D.

Take MA, MC any equimultiples of A, C: and becauſe A, C are equimultiples of B, D: and MA, MC are equimultiples

of A, C; therefore MA, MC are equimultiples of B, D [a];
that is, they contain B, D equally; as many
a 3. 5. times, therefore, as any multiple of A contains B, ſo many times does the ſame multiple of C contain D: Wherefore, as A is to
b 5. Def. 5. B, ſo [b] is C to D.



Next, Let A be the ſame part of B, that C is of D: A is to B, as C to D. For B is the ſame multiple of A, that D is of C; therefore, by the preceding caſe, B is to A, as D to C; and in-
c B. 5. verſely [c], A is to B, as C is to D. Therefore, &c. Q. E. D.

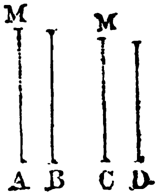
A, B, C, D.

PROP. D. THEOR.

IF the firſt be to the ſecond, as the third to the fourth; and if the firſt be a multiple, or a part of the ſecond; the third is the ſame multiple, or the ſame part of the fourth.

Let A be to B, as C is to D; and let A be a part of B; C is the ſame part of D.

Take MA equal to B, and take MC the ſame multiple of C, that MA or B is of A: and becauſe A is to B, as C to D, and MA, MC are equimultiples of A, C; therefore MA is to B,
a Cor. 4. 5. as MC to D [a]: and the firſt MA is equal to the ſecond B; therefore the third MC is equal to the
b A. 5. fourth D [b]: and C is the ſame part of MC, that A is of B; therefore C is the ſame part of D, that A is of B.



Next, Let A be to B, as C to D; and A a multiple of B; C is the ſame multiple of D.

A, B, C, D.

c B. 5. Becauſe A is to B, as C to D; then, inverſely [c], B is to A, as D to C: but B is a part of A, therefore, by the preceding caſe, D is the ſame part of C; that is, C is the ſame multiple of D, that A is of B. Therefore, &c. Q. E. D.

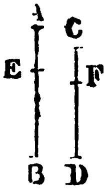
PROP. E. THEOR.

IF four magnitudes be proportionals, they are alſo proportionals by converſion; that is, the firſt is to its exceſs above the ſecond, as the third to its exceſs above the fourth.

Let AB be to BE, as CD to DF; then, by converſion, BA is to AE, as DC to CF.

Becauſe AB is to BE, as CD to DF, by diviſion [a], AE is to EB, as CF to FD; and, by inverſion [b], BE is to EA, as DF to FC. Wherefore, by compoſition [c], BA is to AE, as DC to CF. If, therefore, &c. Q. E. D.

a 17. 5.
b B. 5.
c 18. 5.



PROP. XX. THEOR.

IF there be three magnitudes, and other three, which, taken two and two, have the ſame ratio; if the firſt be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

Let A, B, C be three magnitudes, and D, E, F other three, which, taken two and two, have the ſame ratio, viz. as A is to B, ſo is D to E; and as B to C, ſo is E to F. If A be greater than C, D ſhall be greater than F; and if equal, equal; and if leſs, leſs.

A, B, C,
D, E, F.

Becauſe A is greater than C, and B is any magnitude, A has to B a greater ratio than C has to B [a]: But as D is to E, ſo is A to B; therefore D has to E a greater ratio than C to B [b]: and becauſe B is to C, as E to F, by inverſion [c], C is to B, as F is to E; and D was ſhown to have to E a greater ratio than C to B; therefore D has to E a greater ratio than F to E [d]: But the magnitude which has a greater ratio than another to the ſame magnitude, is the greater of the two [e]; D is therefore greater than F.

a 8. 5.
b 13. 5.
c B. 5.
d Cor. 13. 5.
e 10. 5.

Secondly, Let A be equal to C; D ſhall be equal to F: Becauſe A is equal to C, A is to B, as C is to B [f]: But A is to B, as D to E; and C is to B, as F to E; wherefore D is to E, as F to E [g]; and therefore D is equal to F [h].

f 7. 5.
g 11. 5.
h 9. 5.

Next, Let A be leſs than C; D ſhall be leſs than F: For, as was ſhown in the firſt caſe, C is to B, as F to E, and in like manner, B is to A, as E to D; and C is greater than A, therefore F is greater than D, by the firſt caſe; that is, D is leſs than F. Therefore, &c. Q. E. D.

PROP. XXI. THEOR.

IF there be three magnitudes, and other three, which have the ſame ratio taken two and two, but in a croſs order; if the firſt be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

Let A, B, C be three magnitudes, and D, E, F other three, which have the ſame ratio, taken two and two, but in a croſs order, viz. as A is to B, ſo is E to F, and as B is to C, ſo is D to E. If A be greater than C, D ſhall be greater than F; and if equal; equal, and if leſs, leſs.

Becauſe A is greater than C, A has to B a greater ratio than
a 8. 5. C has to B[a]: But E is to F, as A to B; therefore E has to F
b 13. 5. a greater ratio than C to B[b]: and becauſe B is to
c B. 5. C, as D to E, by inverſion[c], C is to B, as E to D:
and E was ſhown to have to F a greater ratio than
C to B; therefore E has to F a greater ratio than
dCor.13.5. E to D[d]: But the magnitude to which the ſame has a greater
e 10. 5. ratio than it has to another, is the leſſer of the two[e]; F therefore is leſs than D; that is, D is greater than F.

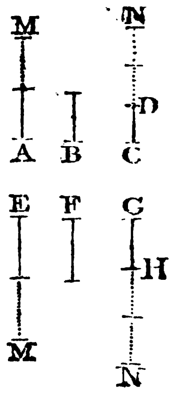
A,	B,	C,
D,	E,	F.

Secondly, Let A be equal to C; D ſhall be equal to F. Be-
f 7. 5. cauſe A and C are equal, A is to B, as C is to B[f]: But A is to
B, as E to F; and C to B, as E to D; wherefore E is to F, as
g 11. 5. E to D[g]; and therefore D is equal to F[h].
h 9. 5. Next, Let A be leſs than C; D ſhall be leſs than F: For, as was ſhown, C is to B, as E to D, and, in like manner, B is to A, as F to E: and C is greater than A; therefore, by caſe firſt, F is greater than D; that is, D is leſs than F: Therefore, &c. Q. E. D.

PROP. XXII. THEOR.

IF there be any number of magnitudes, and as many others, which, taken two and two in order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt; that is, the firſt and laſt ſhall be proportionals by equality.

Firſt, Let there be three magnitudes A, B, CD, and as many others E, F, GH, which, taken two and two, have the ſame ratio, that is, ſuch that A is to B, as E to F; and B to CD, as F to GH: A ſhall be to CD, as E to GH.



Take MA any multiple of A greater than CD, and ME the ſame multiple of E: alſo take NC the leaſt multiple of CD that is greater than MA, and NG the ſame multiple of GH: and becauſe A is to B, as E to F, and that MA, ME are equimultiples of A,
a Cor. 4. 5. E; therefore MA is to B, as ME to F[a]:
and

and becauſe B is to CD, as F to GH, and NC, NG are equimultiples of CD, GH, B is to NC, as F to NG [a]: and becauſe MA is to B, as ME to F, and B to NC, as F to NG; and that MA is leſs than NC; therefore ME is leſs than NG [b]. In like manner, becauſe MA is not leſs than ND, it may be proved, that ME is not leſs than NH: Wherefore MA, ME contain CD, GH the ſame number of times that ND, NH contain them, that is, equally: and MA, ME are any equimultiples of A, E; as many times, therefore, as any multiple of A contains CD, ſo many times does the ſame multiple of E contain GH: Therefore [c] as A is to CD, ſo is E to GH.

a Cor. 4. 5.
b 20. 5.
c 5. Def. 5.

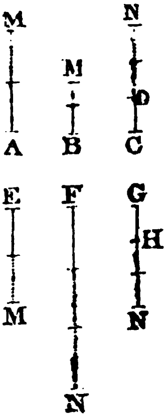
Next, Let there be four magnitudes A, B, C, D, and other four E, F, G, H, which two and two have the ſame ratio, viz. as A is to B, ſo is E to F; as B to C, ſo F to G; and as C to D, ſo G to H: A ſhall be to D, as E to H.

A, B, C, D,
E, F, G, H.

Becauſe A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the ſame ratio; therefore, by the firſt caſe, A is to C, as E to G: But C is to D, as G to H; therefore again, by the firſt caſe, A is to D, as E to H; and ſo on, whatever be the number of magnitudes. Wherefore, &c. Q. E. D.

PROP. XXIII. THEOR.

If there be any number of magnitudes, and as many others, which, taken two and two, in a croſs order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt *.



Firſt, Let there be three magnitudes A, B, CD, and other three E, F, GH, which, taken two and two in a croſs order, have the ſame ratio, that is, ſuch that A is to B, as F to GH, and as B is to CD, ſo is E to F: A ſhall be to CD, as E is to GH.

Take MA, MB, ME any equimultiples of A, B, E, ſo that MA be greater than CD; and take NC the leaſt multiple of CD that is greater than MA, and NF, NG the ſame multiples of F, GH: and becauſe MA, MB are equimultiples of A, B; A is to B, as MA to MB [a]: and, for the ſame reaſon, F is to GH, as NF to NG: But as A is to B, ſo is F to

a 15. 5.

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* N. B. This is ſaid to be by Perturbate Equality.

to GH; as therefore MA is to MB, ſo is NF to NG [b]. And
becauſe B is to CD, as E to F, and that MB, ME are equi-
b 11. 5. multiples of B; E, and NC, NF of CD, F; as MB is to NC,
c 4. 5. ſo is ME to NF [c]: Therefore MA, MB, NC are three mag-
nitudes, and ME, NF, NG other three, which have the ſame
ratio, taken two and two in a croſs order: and MA is leſs than
d 21. 5. NC; therefore ME is leſs than NG [d]: In like manner, be-
cauſe MA is not leſs than ND, it may be proved, that ME is
not leſs than NH; and ND, NH contain CD, GH equally;
therefore MA, ME contain them equally; as many times,
therefore, as any multiple of A contains CD, ſo many times
does the ſame multiple of E contain GH; therefore [e], as A is to
e 5.Def. 5. CD, ſo is E to GH.

Next, Let there be four magnitudes, A, B, C, D, and other four E, F, G, H, which, taken two and two in a croſs order, have the ſame ratio, viz. A to B, as G to H; B to C, as F to G; and C to D, as E to F: A is to D, as E to H.

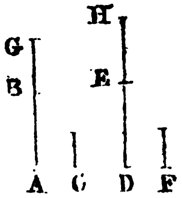
A,	B,	C,	D,
E,	F,	G,	H.

Becauſe A, B, C are three magnitudes, and F, G, H other three, which, taken two and two in a croſs order, have the ſame ratio; by the firſt caſe, A is to C, as F to H: But C is to D, as E to F; wherefore again, by the firſt caſe, A is to D, as E to H: and ſo on, whatever be the number of magnitudes. Wherefore, &c. Q. E. D.

PROP. XXIV. THEOR.

IF the firſt has the ſame ratio to the ſecond, which the third has to the fourth; and the fifth to the ſecond, the ſame ratio that the ſixth has to the fourth; the firſt and fifth together ſhall have to the ſecond, the ſame ratio which the third and ſixth together have to the fourth.

Let AB the firſt, have to C the ſecond, the ſame ratio which DE the third, has to F the fourth; and let BG the fifth, have to C the ſecond, the ſame ratio which EH the ſixth, has to F the fourth; AG, the firſt and fifth together, ſhall have to C the ſecond, the ſame ratio which DH, the third and ſixth together, has to F the fourth.



Becauſe BG is to C, as EH to F; by in-
a B. 5. verſion, [a], C is to BG, as F to EH: and
becauſe,

becauſe, as AB is to C, ſo is DE to F; and as C to BG, ſo F to EH; by equality [b], AB is to BG, as DE to EH: and, by compoſition [c], as AG is to GB, ſo is DH to HE; but as GB is to C, ſo is HE to F; therefore, by equality [b], as AG is to C, ſo is DH to F. Wherefore, &c. Q. E. D.

b 22. 5.
c 18. 5.

Cor. 1. If the ſame hypotheſis be made as in the propoſition, the exceſs of the firſt and fifth ſhall be to the ſecond, as the exceſs of the third and ſixth to the fourth. The demonſtration of this is the ſame with that of the propoſition, if diviſion be uſed inſtead of compoſition.

Cor. 2. The propoſition holds true, of two ranks of magnitudes, whatever be their number, of which each of the firſt rank has to the ſecond magnitude the ſame ratio that the correſponding one of the ſecond rank has to a fourth magnitude; as is manifeſt.

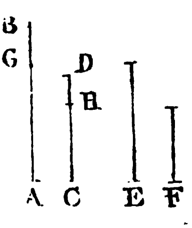
PROP. XXV. THEOR.

If four magnitudes of the ſame kind be proportionals, the greateſt and leaſt of them together are greater than the other two together.

Let AB be to CD, as E is to F, and let AB be the greateſt of them, and conſequently F the leaſt [a]: AB, together with F, is greater than CD, together with E.

a A. & 14. 5.

Take AG equal to E, and CH equal to F: Then, becauſe as AB is to CD, ſo is E to F, and that AG is equal to E, and CH equal to F; AB is to CD, as AG to CH: wherefore the remainder GB is to the remainder DH, as the whole AB to the whole CD [b]: But AB is greater than CD; therefore GB is greater than HD [c]. And becauſe AG is equal to E, and CH to F; AG and F together are equal to CH and E together [d]. If therefore AG and F be added to the greater magnitude BG, and CH and E to the leſs DH; the whole AB and F together are greater than the whole CD and E [e] together. Wherefore, &c. Q. E. D.



b 19. 5.
c A. 5.
d 2. Ax. 1.
e 4. Ax. 1.

PROP.

PROP. F. THEOR.

RATIOS which are compounded of the ſame ratios, are the ſame with one another.

Let the ratios of A to B, and B to C, be the ſame with the ratios of D to E and E to F; the ratio of A to C, which is compounded of the firſt ratios, is the ſame with the ratio of D to F, which is compounded of the other ratios.

A, B, C,
D, E, F.

Firſt, Let A be to B, as D to E; and B to C, as E to F; then, by equality [a], as A is to C, ſo is D to F.

Next, Let A be to B, as E to F; and B to C, as D to E; therefore, by perturbate equality [b], A is to C, as D to F; that is, the ratio compounded of the ratios of A to B, and B to C, is the ſame with the ratio compounded of the ratios of D to E, and E to F: and in like manner, the propoſition may be demonſtrated, whatever be the number of ratios in either caſe.

[a] 22. 5.

[b] 23. 5.

THE

THE

ELEMENTS

OF

EUCLID.

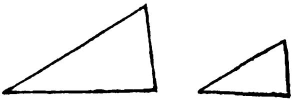
BOOK VI.

DEFINITIONS.

I.

Book VI.

SIMILAR rectilineal figures are thoſe which have their ſeveral angles equal, each to each, and the ſides about the equal angles proportionals.



II.

Two magnitudes are ſaid to be reciprocally proportional to two others, when one of the firſt is to one of the other magnitudes, as the remaining one of the laſt two is to the remaining one of the firſt.

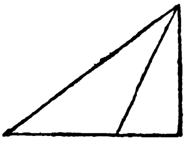
See N.

III.

A ſtraight line is ſaid to be cut in extreme and mean ratio, when the whole is to the greater ſegment, as the greater ſegment is to the leſs.

IV.

The altitude of any figure is the ſtraight line drawn from its vertex perpendicular to the baſe.



PROP.

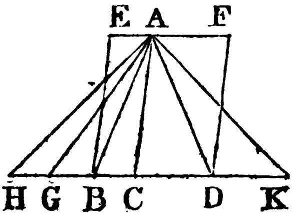
Book VI.

PROP. I. THEOR.

See N. TRiangles and parallelograms of the ſame altitude are one to another as their baſes.

Let the triangles ABC, ACD, and the parallelograms EC, CF have the ſame altitude, viz. the perpendicular drawn from the point A to BD: Then, as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

Produce BD both ways to the points H, K, and take any number of ſtraight lines BG, GH, each equal to the baſe BC, ſo that CH be greater than CD; and join AG, AH: and becauſe the baſes CB, BG, GH are all equal, the triangles ABC, a 38. 1. AGB, AHG are all equal [a]; therefore the triangle AHC is the ſame multiple of ABC, that the baſe HC is of BC. In like manner, if CK be taken the leaſt multiple of CD that is greater than CH, and AK be joined, it may be proved, that the triangle ACK is the ſame multiple of ACD: Wherefore the baſe DK and the triangle ADK contain CD and ACD equally. And if the baſe HC be equal to DK, the triangle AHC is equal [a] to the triangle ADK: But, if not, the baſe HC is greater than DK, but leſs than CK, and the triangle AHC is greater than ADK, but leſs than ACK; therefore the baſe CH contains CD the ſame b Def. A. 5. number of times that DK contains it [b]; and the triangle AHC contains ACD the ſame number of times that ADK does: But the baſe DK and the triangle ADK contain DC and ACD equally; therefore the baſe HC and triangle AHC contain them equally: and the baſe HC and tirangle AHC are any equimultiples of BC and ABO: as many times, therefore, as any multiple of the baſe BC contains the baſe CD, ſo many times does the ſame multiple of the triangle ABC contain the triangle c 5. Def. 5. ACD; therefore [c] as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD.



And becauſe the parallelogram CE is double of the triangle c 41. 1. ABC [c], and the parallelogram CF double of the triangle ACD, and that magnitudes have the ſame ratio which their equimul- d 15. 5. tiples have [d]; as the triangle ABC is to ACD, ſo is the parallelogram EC to CF: And as the baſe BC is to CD, ſo is the triangle ABC to ACD; therefore, as the baſe BC is to CD, ſo is

the

[e] the parallelogram EC to the parallelogram CF. Wherefore triangles, &c. Q. E. D.

Cor. From this it is plain, that triangles and parallelograms e 11. 5. that have equal altitudes, are one to another as their baſes.

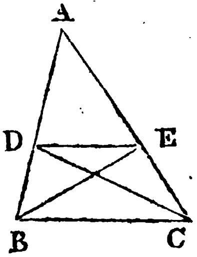
Let the figures be placed ſo as to have their baſes in the ſame ſtraight line; and having drawn perpendiculars from the vertices to the baſes, the ſtraight line which joins the vertices, is parallel to their baſes [f], becauſe the perpendiculars are both equal f 33. 1. and parallel to one another: Then, if the ſame conſtruction be made as in the propoſition, the demonſtration will be the ſame.

PROP. II. THEOR.

If a ſtraight line be drawn parallel to one of the ſides of a triangle, it ſhall cut the other ſides, or thoſe produced, proportionally: And if the ſides, or the ſides produced, be cut proportionally, the ſtraight line which joins the points of ſection, ſhall be parallel to the remaining ſide of the triangle.

Let DE be drawn parallel to BC, one of the ſides of the triangle ABC: BD is to DA, as CE to EA.

Join BE, CD; then the triangle BDE is equal to the triangle CDE [a], becauſe they are on the ſame baſe DE, and be- a 37. 1. tween the ſame parallels DE, BC: ADE is another triangle, and equal magnitudes have to the ſame, the ſame ratio [b]; there- b 7. 5. fore, as the triangle BDE to the triangle ADE, ſo is the triangle CDE to the triangle ADE; but as the triangle BDE to the triangle ADE, ſo is [c] BD to DA, becauſe having the ſame c 1. 6. altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their baſes; and for the ſame reaſon, as the triangle CDE to the triangle ADE, ſo is CE to EA. Therefore, as BD to DA, ſo is CE to EA [d]. d 11. 5.



Next, Let the ſides AB, AC of the triangle ABC, or theſe produced, be cut proportionally in the points D, E, that is, ſo that BD be to DA, as CE to EA, and join DE: DE is parallel to BC.

The ſame conſtruction being made, becauſe, as BD to DA, ſo is CE to EA; and as BD to DA, ſo is the triangle BDE to the triangle ADE [e]; and as CE to EA, ſo e 1. 6. is the triangle CDE to the triangle ADE; therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE; that is, the triangles BDE,

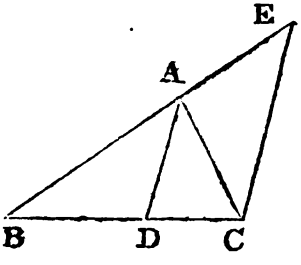
CDE have the ſame ratio to the triangle ADE; and therefore [f]
the triangle BDE is equal to the triangle CDE: And they are
f 9. 5. on the ſame baſe DE; but equal triangles on the ſame baſe are
g 39. 1. between the ſame parallels [g]; therefore DE is parallel to BC.
Wherefore, if a ſtraight line, &c. Q. E. D.

PROP. III. THEOR.

IF the angle of a triangle be divided into two equal angles, by a ſtraight line which alſo cuts the baſe; the ſegments of the baſe ſhall have the ſame ratio which the other ſides of the triangle have to one another: And if the ſegments of the baſe have the ſame ratio which the other ſides of the triangle have to one another, the ſtraight line drawn from the vertex to the point of ſection, divides the vertical angle into two equal angles.

Let the angle BAC of any triangle ABC be divided into two equal angles by the ſtraight line AD: BD is to DC, as BA to AC.

a 31. 1. Through the point C draw CE parallel [a] to DA, and let BA
produced meet CE in E. Becauſe the ſtraight line AC meets
the parallels AD, EC, the angle ACE is equal to the alternate
b 29. 1. angle CAD [b]: But CAD, by the hypotheſis, is equal to the
angle BAD; wherefore BAD is equal to the angle ACE.
Again, becauſe the ſtraight line
BAE meets the parallels AD, EC,
the outward angle BAD is equal
to the inward and oppoſite angle
AEC [b]: But the angle ACE has
been proved equal to the angle
BAD; therefore alſo ACE is equal
to the angle AEC, and conſequently
c 6. 1. the ſide AE is equal to the ſide [c]
AC: And becauſe AD is drawn
parallel to one of the ſides of the triangle BCE, viz. to EC,
d 2. 6. BD is to DC, as BA to AE [d]: But AE is equal to AC; there-
e 7. 5. fore, as BD to DC, ſo is BA to AC [e].



Let now BD be to DC, as BA to AC, and join AD; the angle BAC is divided into two equal angles by the ſtraight line AD.

The ſame conſtruction being made; becauſe, as BD to DC,
ſo is BA to AC; and as BD to DC, ſo is BA to AE [d], becauſe
f 11. 5. AD is parallel to EC; therefore BA is to AC, as BA to AE [f]:
Conſequently

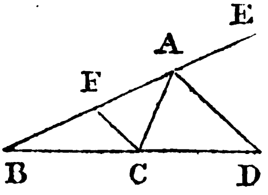
Conſequently AC is equal to AE [g], and the angle AEC is therefore equal to the angle ACE [h]: But the angle AEC is equal to the outward and oppoſite angle BAD; and the angle ACE is g 9. 5. equal to the alternate angle CAD [b]: Wherefore alſo the angle h 5. 1. BAD is equal to the angle CAD: Therefore the angle BAC is cut into two equal angles by the ſtraight line AD. Therefore, if the angle, &c. Q. E. D.

PROP. A. THEOR.

IF the outward angle of a triangle made by producing one of its ſides, be divided into two equal angles, by a ſtraight line which alſo cuts the baſe produced; the ſegments between the dividing line and the extremities of the baſe, have the ſame ratio which the other ſides of the triangle have to one another: And if the ſegments of the baſe produced, have the ſame ratio which the other ſides of the triangle have, the ſtraight line drawn from the vertex to the point of ſection, divides the outward angle of the triangle into two equal angles.

Let the outward angle CAE of any triangle ABC be divided into two equal angles by the ſtraight line AD which meets the baſe produced in D: BD is to DC, as BA to AC.

Through C draw CF parallel to AD [a]; and becauſe the a 31. 1. ſtraight line AC meets the parallels AD, FC, the angle ACF is equal to the alternate angle CAD [b]: But CAD is equal to the b 29. 1. angle DAE [c]; therefore alſo DAE is equal to the angle ACF. c Hyp. Again, becauſe the ſtraight line FAE meets the parallels AD, FC, the outward angle DAE is equal to the inward and oppoſite angle CFA: But the angle ACF has been proved equal to the angle DAE; therefore alſo the angle ACF is equal to the angle CFA, and conſequently the ſide AF is equal to the ſide AC [d]: And be- d 6. 1. cauſe AD is parallel to FC, a ſide of the triangle BCF, BD is to DC, as BA to AF [e]; but AF is e 2. 6. equal to AC; as therefore BD is to DC, ſo is BA to AC.



Let now BD be to DC, as BA to AC, and join AD; the angle CAD is equal to the angle DAE.

The ſame conſtruction being made, becauſe BD is to DC, as BA to AC; and that BD is alſo to DC, as BA to AF [e];

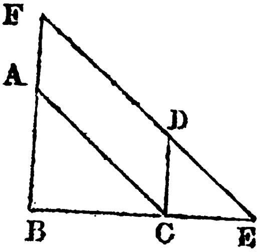
therefore BA is to AC, as BA to AF[f]; wherefore AC is equal
to AF[g], and the angle AFC equal[h] to the angle ACF: But
f 11. 5. the angle AFC is equal to the outward angle EAD, and the
g 9. 5. h 5. 1. angle ACF to the alternate angle CAD; therefore alſo EAD is
equal to the angle CAD. Wherefore, if the outward, &c.
Q. E. D.

PROP. IV. THEOR.

THE ſides about the equal angles of equiangular triangles are proportionals; and thoſe which are oppoſite to the equal angles, are homologous ſides, that is, are the antecedents or conſequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle
ABC equal to the angle DCE, and the angle ACB to the angle
a 32. 1. DEC, and conſequently[a] the angle BAC equal to the angle
CDE. The ſides about the equal angles of the triangles ABC,
DCE are proportionals; and thoſe are the homologous ſides
which are oppoſite to the equal angles.

Let the triangle DCE be placed, ſo that its ſide CE may be
contiguous to BC, and in the ſame ſtraight line with it: And
becauſe the angle ABC is equal to the
b 28. 1. angle DCE, BA is parallel[b] to CD.
Again, becauſe the angle ACB is equal
to the angle DEC, AC is parallel to
DE[b]: Produce BA, ED, until they
c Cor. 39. 1. meet[c] in F: Therefore FACD is a parallelogram; and conſequently AF is
d 34. 1. equal to CD, and AC to FD[d]: And
becauſe AC is parallel to FE, one of
the ſides of the triangle FBE, BA is
e 2. 6. to AF, as BC to CE[e]: But AF is equal to CD; therefore[f],
f 7. 5. as BA to CD, ſo is BC to CE; and alternately, as AB to BC,
g 16. 5. ſo is DC to CE[g]: Again, becauſe CD is parallel to BF, as
BC to CE, ſo is FD to DE[e]; but FD is equal to AC; therefore, as BC to CE, ſo is AC to DE: And alternately[g], as BC
to CA, ſo CE to ED: Therefore, becauſe it has been proved
that AB is to BC, as DC to CE, and as BC to CA, ſo CE to
h 22. 5. ED, by equality[h], BA is to AC, as CD to DE. Therefore
the ſides, &c. Q. E. D.



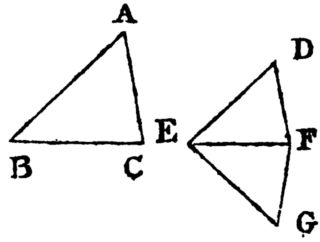
PROP.

PROP. V. THEOR.

IF the ſides of two triangles, about each of their angles, be proportionals, the triangles ſhall be equiangular, and have their equal angles oppoſite to the homologous ſides.

Let the triangles ABC, DEF have their ſides proportionals, ſo that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and conſequently, by equality, BA to AC, as ED to DF; the triangle ABC is equiangular to the triangle DEF, and their equal angles are oppoſite to the homologous ſides, viz. the angle ABC equal to the angle DEF, and BCA to EFD, and alſo BAC to EDF.

At the points E, F, in the ſtraight line EF, make [a] the angle FEG equal to the angle ABC, and the angle EFG equal to BCA; wherefore the remaining angle BAC is equal to the remaining angle EGF [b], and the triangle ABC is therefore equiangular to the triangle GEF; and conſequently they have their ſides oppoſite to the equal angles proportionals [c]: Wherefore, as AB to BC, ſo is GE to EF; but as AB to BC, ſo is DE to EF; therefore as DE to EF, ſo [d] GE to EF: Therefore DE and GE have the ſame ratio to EF, and conſequently are equal [e]: For the ſame reaſon, DF is equal to FG: And becauſe, in the triangles DEF, GEF, DE is equal to EG, and EF common, the two ſides DE, EF are equal to the two GE, EF, and the baſe DF is equal to the baſe GF; therefore the angle DEF is equal [f] to the angle GEF: and GEF is equal to the angle ABC; therefore the angle ABC is equal to the angle DEF: For the ſame reaſon, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if the ſides, &c. Q. E. D.



a 23. 1.

b 32. 1.

c 4. 1.

d 11. 5.

e 9. 5.

f 8. 1.

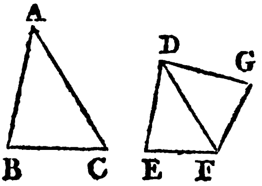
PROP.

PROP. VI. THEOR.

IF two triangles have one angle of the one equal to one angle of the other, and the ſides about the equal angles proportionals, the triangles ſhall be equiangular, and ſhall have thoſe angles equal which are oppoſite to the homologous ſides.

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the ſides about thoſe angles proportionals; that is, BA to AC, as ED to DF; the triangles ABC, DEF are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

a 23. 1. At the points D, F, in the ſtraight line DF, make [a] the
angle FDG equal to either of the angles BAC, EDF; and the
angle DFG equal to the angle ACB:
Wherefore the remaining angle at B
b 32. 1. is equal to the remaining one at G [b],
and conſequently the triangle ABC
is equiangular to the triangle DGF;
c 4. 6. and therefore, as BA to AC, ſo is [c]
GD to DF: But, by the hypotheſis,
as BA to AC, ſo is ED to DF; as,
d 11. 5. therefore, ED to DF, ſo is [d] GD to DF; wherefore ED is
e 9. 5. equal [e] to DG; and DF is common to the two triangles EDF,
GDF: Therefore the two ſides ED, DF are equal to the two
ſides GD, DF; and the angle EDF is equal to the angle
f 4. 1. GDF; wherefore the baſe EF is equal to the baſe FG [f], and
the angle DFG to the angle DFE: But the angle DFG is
equal to the angle ACB; therefore the angle ACB is equal
to the angle DFE: And the angle BAC is equal to the
g Hyp. angle EDF [g]; wherefore alſo the remaining angle at B is
equal to the remaining angle at E. Therefore the triangle
ABC is equiangular to the triangle DEF. Wherefore, if two
triangles, &c. Q. E. D.

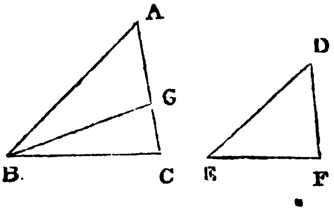


PROP. VII. THEOR.

See N.

IF two triangles have one angle of the one equal to one angle of the other, and the ſides about two other angles proportionals, then, if either one of the remaining angles be a right angle, or the two together be not equal to two right angles: The triangles ſhall be equiangular, and have thoſe angles equal about which the ſides are proportionals.

Let the two triangles ABC, DEF have one angle BAC of the one equal to one angle EDF of the other, and the ſides about two other angles ABC, DEF proportionals, ſo that AB is to BC, as DE to EF; and let either one of the remaining angles at C, F be a right angle, or elſe the two together not be equal to two right angles: The triangle ABC is equiangular to the triangle DEF, viz. the angle ABC is equal to the angle DEF, and the remaining angle at C, to the remaining angle at F.



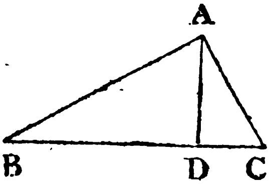
For, if the angles ABC, DEF be not equal, one of them is greater than the other; let ABC be the greater, and at the point B, in AB, make the angle ABG equal to DEF [a]: And becauſe the angle at A is equal a 23. 1. to the angle at D, and the angle ABG to the angle DEF; the remaining angle AGB is equal [b] to the remaining angle DFE: Therefore the triangle b 32. 1. ABG is equiangular to the triangle DEF; wherefore, [c] as AB is to BG, ſo is DE to EF; c 4. 6. but as DE to EF, ſo, by hypotheſis, is AB to BC; therefore, as AB to BC, ſo is AB to BG [d]: and becauſe AB has the ſame ratio to each of the lines BC, BG; d 11. 5. BC is equal to BG [e], and therefore the angle BGC is equal to BCG [f]: Wherefore each of the angles BCG, BGC is leſs than a e 9. 5. right angle [g], and conſequently AGB is greater than a right f 5. 1. angle [h]: But it was proved, that the angle AGB is equal to g 17. 1. DFE; therefore DFE is greater than a right angle: and be- h 13. 1. cauſe, by hypotheſis, the angles ACB, DFE are not equal to two right angles, and that CGB is equal to ACB, and AGB to DFE; therefore the angles AGB, BGC are not equal to two right angles: But they are equal [h] to two right angles, becauſe they are adjacent angles; which is impoſſible.

Therefore the angles ABC, DEF are not unequal, that i they are equal; and the angle at A is equal to the angle at D therefore the remaining angle at C is equal to the remaini angle at F: Wherefore the triangle ABC is equiangular to t triangle DEF. Therefore, &c. Q. E. D.

PROP. VIII. THEOR.

IN a right angled triangle, if a perpendicular b drawn from the right angle to the baſe; the triangles on each ſide of it are ſimilar to the whole triangle, and to one another.

Let ABC be a right angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the baſe BC: The triangles ABD, ADC are ſimilar to the whole triangle ABC, and to one another.



Becauſe the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD; the remaining angle ACB is equal to
a 32. 1. the remaining angle BAD [a]: Therefore the triangle ABC is equiangular to the triangle ABD, and the ſides about their equal angles are pro-
b 4. 6. portionals [b]; wherefore the triangles
c 1. Def. 6. are ſimilar [c]: In the like manner, it may be demonſtrated, that the triangle ADC is equiangular and ſimilar to the triangle ABC: And the triangles ABD, ADC, being equiangular to ABC, are equiangular, and therefore ſimilar [b] to each other. Therefore, in a right angled, &c. Q. E. D.

Cor. From this it is manifeſt, that the perpendicular drawn from the right angle of a right angled triangle to the baſe, is a mean proportional between the ſegments of the baſe: And alſo that each of the ſides is a mean proportional between the baſe, and its ſegment adjacent to that ſide: Becauſe in the triangles BDA, ADC, BD is to DA, as DA to DC [b]; and in the triangles ABC, DBA, BC is to BA, as BA to BD [b]; and in the triangles ABC, ACD, BC is to CA, as CA to CD [b].

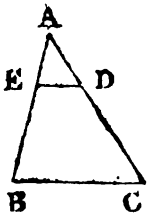
PROP.

PROP. IX. PROB.

FROM a given ſtraight line to cut off any part required.

Let AB be the given ſtraight line; it is required to cut off any part from it.

From the point A draw a ſtraight line AC making any angle with AB; and in AC take any point D, and take AC the ſame multiple of AD, that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it: Then AE is the part required to be cut off.



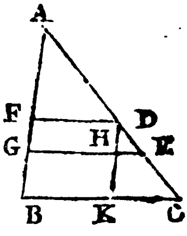
Becauſe ED is parallel to one of the ſides of the triangle ABC, viz. to BC, as CD is
to DA, ſo is [a] BE to EA; and, by compo- a 2. 6.
ſition [b], CA is to AD, as BA to AE: But b 18. 5.
CA is a multiple of AD; therefore [c] BA is c D. 5.
the ſame multiple of AE: Whatever part, therefore, AD is of AC, AE is the ſame part of AB: Wherefore, from the ſtraight line AB the part required is cut off. Which was to be done.

PROP. X. PROB.

TO divide a given ſtraight line ſimilarly to a given divided ſtraight line, that is, into parts that ſhall have the ſame ratios to one another which the parts of the divided given ſtraight line have.

Let AB be the ſtraight line given to be divided, and AC the divided line; it is required to divide AB ſimilarly to AC.

Let AC be divided in the points D, E; and let AB, AC be placed ſo as to contain any angle, and join BC, and through the
points D, E draw [a] DF, EG parallels to it; and through D a 31. 1.
draw DHK parallel to AB: Therefore each of the figures FH, HB is a parallelogram; wherefore DH is
equal [b] to FG, and HK to GB: And becauſe b 34. 1.
HE is parallel to KC, one of the ſides of the
triangle DKC, as CE to ED, ſo is [c] KH to c 2. 6.
HD: But KH is equal to BG, and HD to GF; therefore, as CE to ED, ſo is BG to GF: Again, becauſe FD is parallel to EG, one of the ſides of the triangle AGE, as ED to DA, ſo is GF to FA: But it has been

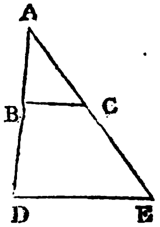


proved, that CE is to ED, as BG to GF; and as ED to DA, ſo GF to FA: Therefore the given ſtraight line AB is divided ſimilarly to AC. Which was to be done.

PROP. XI. PROB.

TO find a third proportional to two given ſtraight lines.

Let AB, AC be the two given ſtraight lines, and let them be placed ſo as to contain any angle; it is required to find a third proportional to AB, AC.



Produce AB, AC to the points D, E; and make BD equal to AC; and having joined
a 31. 1. BC, through D, draw DE parallel to it [a].

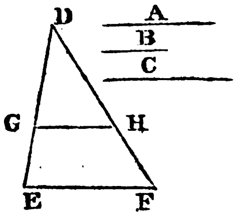
Becauſe BC is parallel to DE, a ſide of the
b 2. 6. triangle ADE, AB is [b] to BD, as AC to CE: But BD is equal to AC; as, therefore, AB to AC, ſo is AC to CE. Wherefore to the two given ſtraight lines AB, AC a third proportional CE is found. Which was to done.

PROP. XII. PROB.

TO find a fourth proportional to three given ſtraight lines.

Let A, B, C be the three given ſtraight lines; it is required to find a fourth proportional to A, B, C.

Take two ſtraight lines DE, DF, containing any angle EDF; and upon theſe make DG equal to A, GE equal to B, and DH equal to C; and having joined GH,
a 31. 1. draw EF parallel [a] to it through the point E: And becauſe GH is parallel to EF, one of the ſides of the triangle
b 2. 6. DEF, DG is to GE, as DH to HF [b]; but DG is equal to A, GE to B, and DH to C; therefore, as A is to B, ſo is C to HF. Wherefore to the three given ſtraight lines, A, B, C a fourth proportional HF is found. Which was to be done.

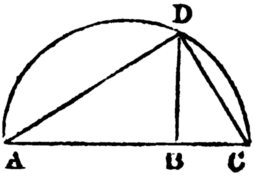


PROP. XIII. PROB.

TO find a mean proportional between two given ſtraight lines.

Let AB, BC be the two given ſtraight lines; it is required to find a mean proportional between them.

Place AB, BC in a ſtraight line, and upon AC deſcribe the ſemicircle ADC, and from the point B draw [a] BD at right angles to AC, and join AD, DC.



Then the angle ADC in a ſemicircle is a right angle [b], and becauſe in the right angled triangle ADC, DB is drawn from the right angle, perpendicular to the baſe, DB is a mean proportional between AB, BC the ſegments of the baſe [c]: Therefore between the two given ſtraight lines AB, BC, a mean proportional DB is found. Which was to be done.

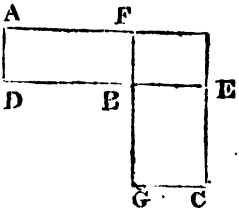
a 11. 1.
b 31. 3.
c Cor. 8. 6.

PROP. XIV. THEOR.

EQUAL parallelograms which have one angle of the one equal to one angle of the other, have their ſides about the equal angles reciprocally proportional: And parallelograms that have one angle of the one equal to one angle of the other, and their ſides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms, which have the angles at B equal, and let the ſides DB, BE be placed in the ſame ſtraight line; wherefore alſo FB, BG are in one ſtraight line [a]: The ſides of the parallelograms AB, BC about the equal angles, are reciprocally proportional; that is, DB is to BE, as GB to BF.

a 3. Cor. 15. 1.



Complete the parallelogram FE; and becauſe the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE, as BC to FE [b]: But as AB to FE, ſo is the baſe DB to BE [c]; and, as BC to FE, ſo is the baſe GB to BF; therefore, as DB to BE, ſo is GB to BF [d]. Wherefore the ſides

b 7. 5.
c 1. 6.
d 11. 5.

of the parallelograms AB, BC about their equal angles are reciprocally proportional.

But, let the ſides about the equal angles be reciprocally proportional, viz. as DB to BE, ſo GB to BF; the parallelogram AB is equal to the parallelogram BC.

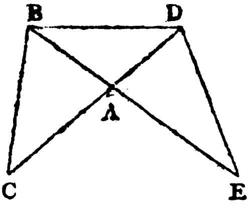
Becauſe, as DB is to BE, ſo GB to BF; and as DB to BE,
c 1. 6. ſo [c] is the parallelogram AB to the parallelogram FE; and as
GB to BF, ſo is the parallelogram BC to the parallelogram FE;
d 11. 5. therefore, as AB to FE, ſo BC to FE [d]: Wherefore the paralle-
e 9. 5. logram AB is equal [e] to the parallelogram BC. Therefore equal
parallelograms, &c. Q. E. D.

PROP. XV. THEOR.

EQUAL triangles which have one angle of the one equal to one angle of the other, have their ſides about the equal angles reciprocally proportional: And triangles which have one angle in the one equal to one angle in the other, and their ſides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE be equal triangles, which have the angle BAC equal to the angle DAE; the ſides about the equal angles of the triangles are reciprocally proportional; that is, CA is to AD, as EA to AB.

Let the triangles be placed ſo that their ſides CA, AD be
in one ſtraight line; wherefore alſo EA and AB are in one
a 3. Cor. 15. 1. ſtraight line [a]; and join BD. Becauſe the triangle ABC is
equal to the triangle ADE, and that
ABD is another triangle; therefore, as
the triangle CAB is to the triangle
BAD, ſo is triangle EAD to triangle
b 7. 5. DAB [b]: But as triangle CAB to
triangle BAD, ſo is the baſe CA to
c 1. 6. AD [c]; and as triangle EAD to triangle
DAB, ſo is the baſe EA to AB [c]; as,
d 11. 5. therefore, CA to AD, ſo is EA to AB [d]:
wherefore the ſides of the triangles ABC, ADE about the equal
angles are reciprocally proportional.



But let the ſides of the triangles ABC, ADE about the equal angles be reciprocally proportional, viz. CA to AD, as EA to AB; the triangle ABC is equal to the triangle ADE.

Having joined BD as before; becauſe, as CA to AD, ſo is

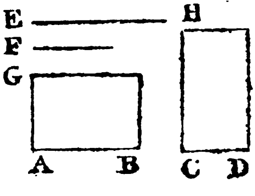
EA to AB; and as CA to AD, ſo is triangle ABC to triangle
BAD [c]; and as EA to AB, ſo is triangle EAD to triangle
BAD [c]; therefore [d] as triangle BAC to triangle BAD, ſo is c 1. 6.
triangle EAD to triangle BAD; that is, the triangles BAC, d 11. 6.
EAD have the ſame ratio to the triangle BAD: Wherefore the
triangle ABC is equal [e] to the triangle ADE. Therefore equal e 9. 5.
triangles, &c. Q. E. D.

PROP. XVI. THEOR.

IF four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: And if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four ſtraight lines are proportionals.

Let the four ſtraight lines, AB, CD, E, F be proportionals, viz. as AB to CD, ſo E to F; the rectangle contained by AB, F is equal to the rectangle contained by CD, E.

From the points A, C draw [a] AG, CH at right angles to a 11. 1.
AB, CD; and make AG equal to F, and CH equal to E, and
complete the parallelograms BG, DH: Becauſe, as AB to CD,
ſo is E to F; and that E is equal to CH, and F to AG; AB
is [b] to CD, as CH to AG: Therefore the ſides of the paralle- b 7. 5.
lograms BG, DH about the equal angles are reciprocally pro-
portional; but parallelograms which have their ſides about
equal angles reciprocally proportional, are equal to one another [c]; c 14. 6.
therefore the parallelogram BG is equal to the parallelogram DH: And the parallelogram BG is contained by the ſtraight lines AB, F; becauſe AG is equal to F; and the parallelogram DH is contained by CD and E; becauſe CH is equal to E: Therefore the rectangle contained by the ſtraight lines AB, F is equal to that which is contained by CD and E.



And if the rectangle contained by the ſtraight lines AB, F be equal to that which is contained by CD, E; theſe four lines are proportionals, viz. AB is to CD, as E to F.

The ſame conſtruction being made, becauſe the rectangle contained by the ſtraight lines AB, F is equal to that which is contained by CD, E, and that the rectangle BG is contained by AB, F, becauſe AG is equal to F; and the rectangle DH by CD, E, becauſe CH is equal to E; therefore the parallelo-

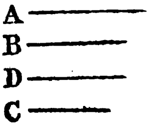
gram BG is equal to the parallelogram DH; and they are equiangular: But the ſides about the equal angles of equal
c 14. 6. parallelograms are reciprocally proportional [c]: Wherefore, as AB to CD, ſo is CH to AG; and CH is equal to E, and AG to F: As, therefore, AB is to CD, ſo E to F. Wherefore, if four, &c. Q. E. D.

PROP. XVII. THEOR.

IF three ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the ſquare of the mean: And if the rectangle contained by the extremes be equal to the ſquare of the mean, the three ſtraight lines are proportionals.

Let the three ſtraight lines A, B, C be proportionals, viz. as A to B, ſo B to C; the rectangle contained by A, C is equal to the ſquare of B.

Take D equal to B; and becauſe as A to B, ſo B to C, and
a 7. 5. that B is equal to D; A is [a] to B, as D to C: But if four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to that which is
b 16. 6. contained by the means [b]: Therefore the rectangle contained by A, C is equal to that contained by B, D: But the rectangle contained by B, D is the ſquare of B; becauſe B is equal to D: Therefore the rectangle contained by A, C is equal to the ſquare of B.



And if the rectangle contained by A, C be equal to the ſquare of B; A is to B, as B to C.

The ſame conſtruction being made, becauſe the rectangle contained by A, C is equal to the ſquare of B, and the ſquare of B is equal to the rectangle contained by B, D, becauſe B is equal to D; therefore the rectangle contained by A, C is equal to that contained by B, D: But if the rectangle contained by the extremes be equal to that contained by the means, the four ſtraight lines are proportionals [b]: Therefore A is to B, as D to C: But B is equal to D; wherefore, as A to B, ſo B to C. Therefore, if three ſtraight lines, &c. Q. E. D.

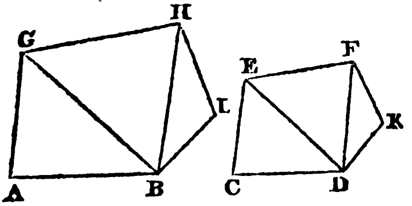
PROP. XVIII. PROB.

See N.

UPON a given ſtraight line, to deſcribe a rectilineal figure ſimilar, and ſimilarly ſituated to a given rectilineal figure.

Let AB be the given ſtraight line, upon which it is required to deſcribe a rectilineal figure ſimilar, and ſimilarly ſituated to one given.

Firſt, Let the given rectilineal be the triangle CDE: And at the points A, B in the ſtraight line AB, make [a] the angle BAG equal to a 23. 1. DCE, and the angle ABG equal to CDE; therefore the remaining angle AGB is equal [b] to the remaining angle CED: Wherefore b 32. 1. the triangle AGB is equiangular, and therefore ſimilar [c] to the triangle CDE. c 4. 6.



Next, Let the given figure be the quadrilateral CDFE: and join DE, and upon AB deſcribe the triangle AGB ſimilar, and ſimilarly ſituated to the triangle CED: and let the ſide GB be homologous to DE: and upon GB deſcribe the triangle BGH ſimilar, and ſimilarly ſituated to the triangle DEF: and becauſe the angle AGB is equal to CED, and BGH to DEF; the whole angle AGH is equal to the whole CEF. For the ſame reaſon, the angle ABH is equal to CDF: and the angles at A, H are equal to thoſe at C, F; therefore the figures AH, CF are equiangular: and becauſe the triangles ABG, CDE are equiangular, AG is to GB, as [c] CE to ED: and becauſe the triangles BGH, DEF are equiangular, BG is to GH, as [c] DE to EF; therefore, by equality [d], AG is to GH, as CE to EF. For the ſame d 22. 5. reaſon, AB is to BH, as CD to DF: and BA is to AG, as DC to CE; and GH to HB, as EF to FD. Wherefore, becauſe the figures AH, CF are equiangular, and have the ſides about their equal angles proportionals, they are ſimilar [e]. e 1. Def. 6.

Again, Let the given figure be CDKFE of five ſides.

Join DF, and upon AB deſcribe the quadrilateral ABHG ſimilar, and ſimilarly ſituated to CDFE: and upon BH deſcribe the triangle BHL ſimilar, and ſimilarly ſituated to DKF: Then, it may be demonſtrated as before, that the rectilineal ABLHG is ſimilar to CDKFE: and ſo on. Which was to be done.

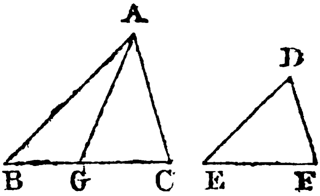
PROP.

PROP. XIX. THEOR.

Similar triangles are to one another in the duplicate ratio of their homologous ſides.

Let ABC, DEF be ſimilar triangles, having the angle B equal to the angle E, and let AB be to BC, as DE to EF, ſo
a 12. Def. 5. that the ſide BC is homologous to EF [a]: The triangle ABC has to the triangle DEF, the duplicate ratio of that which BC has to EF.

b 11. 6. Take BG a third proportional to BC, EF [b], ſo that BC is to EF, as EF to BG, and join GA: Then, becauſe as AB to BC,
c 16. 5. ſo DE to EF; alternately [c], AB is to DE, as BC to EF: But
d 11. 5. as BC to EF, ſo is EF to BG; therefore [d] as AB to DE, ſo is EF to BG: Wherefore the ſides of the triangles ABG, DEF, which are about the equal angles, are reciprocally proportional: But triangles which have the ſides about two equal angles reciprocally proportional, are equal to
e 15. 6. one another [e]: Therefore the triangle ABG is equal to the triangle DEF: And becauſe, as BC is to EF, ſo EF to BG; and that if three ſtraight lines be proportio-
f 10. Def. 5. nals, the firſt is ſaid [f] to have to the third the duplicate ratio of that which it has to the ſecond; BC therefore has to BG the duplicate ratio of that which BC has to EF: But as BC to BG, ſo
g 1. 6. is [g] the triangle ABC to the triangle ABG. Therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF: But the triangle ABG is equal to the triangle DEF; wherefore alſo the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore ſimilar triangles, &c. Q. E. D.



Cor. From this it is manifeſt, that if three ſtraight lines be proportionals, as the firſt is to the third, ſo is any triangle upon the firſt to a ſimilar, and ſimilarly deſcribed triangle upon the ſecond.

PROP.

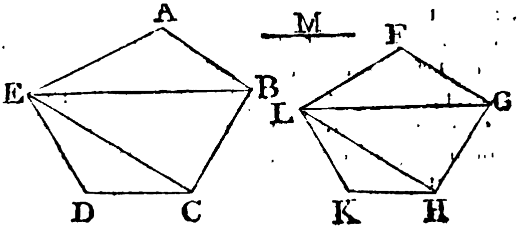
PROP. XX. THEOR.

SIMILAR polygons may be divided into the ſame number of ſimilar triangles, having the ſame ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous ſides have.

Let ABCDE, FGHKL be ſimilar polygons, and let AB be the homologous ſide to FG: The polygons ABCDE, FGHKL may be divided into the ſame number of ſimilar triangles, whereof each to each has the ſame ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the ſide FG.

Join BE, EC, GL, LH: And becauſe the polygon ABCDE is ſimilar to FGHKL, the angle BAE is equal to GFL [a], and BA is to AE, as GF to FL [a]: Wherefore, becauſe the triangles ABE, FGL have an angle in one equal to an angle in the other, and their ſides about theſe angles proportionals, the triangle ABE is equiangular [b], and therefore ſimilar to FGL [c]; wherefore the angle ABE is equal to FGL: And, becauſe the polygons are ſimilar, the whole angle ABC is equal [a] to FGH; therefore the remaining angle EBC is equal to LGH: And becauſe the triangles ABE, FGL are ſimilar, EB is to BA, as LG to GF [a]; and alſo, becauſe the polygons are ſimilar, AB is to BC, as FG to GH [a]; therefore, by equality [d], EB is to BC, as LG to GH; that is, the ſides about the equal angles EBC, LGH are proportionals; therefore [d] the triangle EBC is equiangular to LGH, and ſimilar to it [c]. For the ſame reaſon, the triangle ECD is ſimilar to LHK: Therefore the ſimilar polygons ABCDE, FGHKL are divided into the ſame number of ſimilar triangles.

a 1. Def. 6.
b 6. 6.
c 4. 6.
d 22. 5.



Alſo theſe triangles have, each to each, the ſame ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the conſequents FGL, LGH, LHK: And the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the homologous ſide FG.

Becauſe the triangle ABE is ſimilar to FGL, ABE has to FGL the duplicate ratio [e] of that which the ſide BE has to GL: For the ſame reaſon, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL: Therefore, as the triangle ABE to FGL, ſo [f] is the triangle BEC to GLH. Again, becauſe the triangle EBC is ſimilar to LGH, EBC has to LGH the duplicate ratio of that which EC has to LH: For the ſame reaſon, the triangle ECD has to LHK the duplicate ratio of that which EC has to LH: As, therefore, the triangle EBC to LGH, ſo is [f] ECD to LHK: But it has been proved, that the triangle EBC is to LGH, as ABE to FGL; therefore, as ABE to FGL, ſo is EBC to LGH, and ECD to LHK: and as one of the antecedents to its conſequent, ſo are all the antecedents to all the conſequents [g]. Wherefore, as the triangle ABE to FGL, ſo is the polygon ABCDE to FGHKL: But the triangle ABE has to FGL the duplicate ratio of that which the ſide AB has to the homologous ſide FG. Therefore alſo the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous ſide FG. Wherefore, ſimilar polygons, &c. Q. E. D.

e 19. 6.
f 11. 5.
f 11. 5.
g 12. 5.

Cor. 1. In like manner, it may be proved, that ſimilar four-ſided figures, or of any number of ſides, are one to another in the duplicate ratio of their homologous ſides; and it has already been proved in triangles. Therefore, univerſally ſimilar rectilineal figures are to one another in the duplicate ratio of their homologous ſides.

Cor. 2. And if to AB, FG, two of the homologous ſides, a third proportional M be taken, AB has [h] to M the duplicate ratio of that which AB has to FG: But the four-ſided figure or polygon upon AB has to the four-ſided figure or polygon upon FG likewiſe the duplicate ratio of that which AB has to FG: Therefore, as AB is to M, ſo is the figure upon AB to the figure upon FG; which was alſo proved in triangles [i]. Therefore, univerſally, it is manifeſt, that if three ſtraight lines be proportionals, as the firſt is to the third, ſo is any rectilineal figure upon the firſt, to a ſimilar and ſimilarly deſcribed figure upon the ſecond.

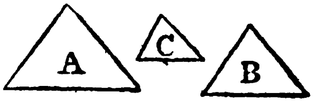
h 10. Def. 5.
i Cor. 19. 6.

PROP. XXI. THEOR.

RECTILINEAL figures which are ſimilar to the ſame rectilineal figure, are alſo ſimilar to one another.

Let each of the rectilineal figures A, B be ſimilar to the rectilineal figure C: The figure A is ſimilar to the figure B.

Becauſe A is ſimilar to C, they are equiangular, and alſo have their ſides about the equal angles proportionals [a]. Again, becauſe B is ſimilar to C, they are equiangular, and have their ſides about the equal angles proportionals [a]: Therefore the figures A, B are each of them equiangular to C, and have the ſides about the equal angles of each of them and of C proportionals. Wherefore the rectilineal figures A and B are equiangular [b], and have their ſides about the equal angles proportionals [c]. Therefore A is ſimilar [a] to B. Q. E. D.



a 1. Def. 6.
b 1. Ax. 1.
c 11. 5.

PROP. XXII. THEOR.

IF four ſtraight lines be proportionals, the ſimilar rectilineal figures ſimilarly deſcribed upon them, ſhall alſo be proportionals; and if the ſimilar rectilineal figures ſimilarly deſcribed upon four ſtraight lines be proportionals, thoſe ſtraight lines ſhall be proportionals.

Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz. AB to CD, as EF to GH, and upon AB, CD let the ſimilar rectilineal figures KAB, LCD be ſimilarly deſcribed; and upon EF, GH the ſimilar rectilineal figures MF, NH, in like manner: The rectilineal figure KAB is to LCD, as MF to NH.

To AB, CD take a third proportional [a] X; and to EF, GH a third proportional O: And becauſe AB is to CD, as EF to GH, and that CD is [b] to X, as GH to O; wherefore, by equality [c], as AB to X, ſo EF to O: But as AB to X, ſo is [d] the rectilineal KAB to the rectilineal LCD, and as EF to O, ſo is [d] the rectilineal MF to the rectilineal NH: Therefore, as KAB to LCD, ſo [b] is MF to NH.

a 11. 6.
b 11. 5.
c 22. 5.
d 2. Cor. 20. 6.
d 2. Cor. 20. 6.

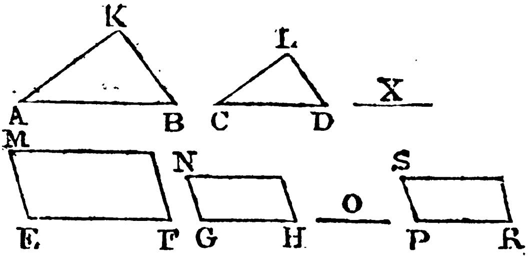
And if the rectilineal KAB be to LCD, as MF to NH; the ſtraight line AB is to CD, as EF to GH.

b 11. 5.

Make [e] as AB to CD, ſo EF to PR, and upon PR deſcribe [f] the rectilineal figure SR ſimilar and ſimilarly ſituated to either of the figures MF, NH: Then, becauſe as AB to CD, ſo is EF to PR, and that upon AB, CD are deſcribed the ſimilar and ſimilarly ſituated rectilineals KAB, LCD, and upon EF, PR, in like manner, the ſimilar rectilineals MF, SR; KAB is to

e 12. 6.
f 18. 6.

LCD, as MF to SR; but, by hypotheſis, KAB is to LCD, as MF to NH; and therefore the rectilineal MF having the
g 9. 5. ſame ratio to each of the two NH, SR, theſe are equal [g] to one



another: They are alſo ſimilar, and ſimilarly ſituated; therefore GH is equal to PR: And becauſe as AB to CD, ſo is EF to PR, and that PR is equal to GH; AB is to CD, as EF to GH. If therefore four ſtraight lines, &c. Q. E. D.

PROP. XXIII. THEOR.

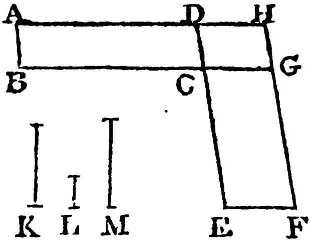
EQUIANGULAR parallelograms have to one another the ratio which is compounded of the ratios of their ſides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG: The ratio of the parallelogram AC to the parallelogram CF, is the ſame with the ratio which is compounded of the ratios of their ſides.

Let BC, CG be placed in a ſtraight line; therefore DC and
a 3. Cor. CE are alſo in a ſtraight line [a]; and complete the parallelogram
15. 1. DG; and, taking any ſtraight line K, make [b] as BC to CG,
b 12. 6. ſo K to L; and as DC to CE, ſo make [b] L to M: Therefore
the ratios of K to L, and L to M, are the ſame with the ratios
of the ſides, viz. of BC to CG, and DC to CE. But the ra-
c A.Def.5 tio of K to M is that which is ſaid to be compounded [c] of the
ratios of K to L, and L to M: Wherefore alſo K has to M,
the ratio compounded of the ratios of the ſides: And becauſe
as BC to CG, ſo is the parallelogram AC to the parallelogram
d 1. 6. CH [d]; but as BC to CG, ſo is K to L; therefore K is [e] to L,
e 11. 5. as the parallelogram AC to the parallelogram CH: Again,
becauſe as DC to CE, ſo is the parallelogram CH to the
parallelogram CF; but as DC to CE, ſo is L to M; wherefore L is [e] to M, as the parallelogram CH to the parallelogram CF:

CF: Therefore, ſince it has been proved, that as K to L, ſo is the parallelogram AC to the parallelogram CH; and as L to M, ſo the parallelogram CH to the parallelogram CF; by equality [f], K is to M, as the parallelogram AC to the parallelogram CF: But K has to M the ratio which is compounded of the ratios of the ſides; therefore alſo the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the ſides. Wherefore, equiangular parallelograms, &c. Q. E. D.

f 22. 5.



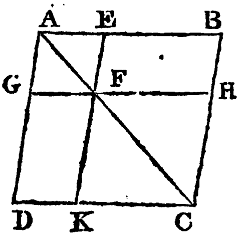
PROP. XXIV. THEOR.

THE parallelograms about the diameter of any parallelogram, are ſimilar to the whole, and to one another.

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK the parallelograms about the diameter: The parallelograms EG, HK are ſimilar both to the whole parallelogram ABCD, and to one another.

Becauſe DC, GF are parallels, the angle ADC is equal [a] to the angle AGF: For the ſame reaſon, becauſe BC, EF are parallels, the angle ABC is equal to the angle AEF: And each of the angles BCD, EFG is equal to the oppoſite angle DAB [b], and therefore are equal to one another; wherefore the parallelograms ABCD, AEFG are equiangular: And becauſe the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC, EAF, they are equiangular to one another; therefore, [c] as AB to BC, ſo is AE to EF: And becauſe the oppoſite ſides of parallelograms are equal to one another [b], AB is [d] to AD, as AE to AG; and DC to CB, as GF to FE; and alſo CD to DA, as FG to GA: Therefore the ſides of the parallelograms ABCD, AEFG about the equal angles are proportionals; and they are therefore ſimilar to one another [e]: For the ſame reaſon, the parallelogram ABCD is ſimilar to the parallelogram FHCK. Wherefore each of the parallelograms GE, KH is ſimilar to DB: But rectilineal figures which are ſimilar to the ſame

a 29. 1.
b 34. 1.
c 4. 6.
d 7. 5.
e 1.Def. 6.



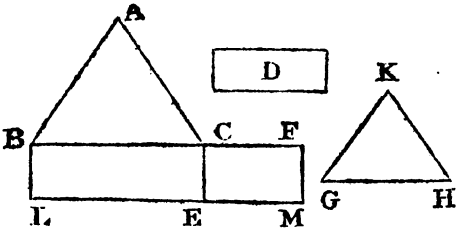
ſame rectilineal figure, are alſo ſimilar to one another [f]; therefore the parallelogram GE is ſimilar to KH. Wherefore the
f 21. 6. parallelograms, &c. Q. E. D.

PROP. XXV. PROB.

TO deſcribe a rectilineal figure which ſhall be ſimilar to one, and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure, to which the figure to be deſcribed is required to be ſimilar, and D that to which it muſt be equal; it is required to deſcribe a rectilineal figure ſimilar to ABC, and equal to D.

a Cor. 45. 1. Upon the ſtraight line BC deſcribe [a] the parallelogram BE
equal to the figure ABC; and produce BC to F, and to CE
apply [a] the parallelogram CM equal to D, and having FCE for
one of its angles: Therefore LE and EM are in a ſtraight
b 29. 1. 14. 1. line [b]: Between BC and CF find [c] a mean proportional GH,
c 13. 6. and upon GH deſcribe [d] the rectilineal figure KGH ſimilar and
d 18. 6. ſimilarly ſituated to the figure ABC: And becauſe BC is to
GH, as GH to CF, and if three ſtraight lines be proportionals,
e 2. Cor. as the firſt is to the third, ſo is [e] the figure upon the firſt to the
20. 6. ſimilar and ſimilarly deſcribed figure upon the ſecond; therefore,
as BC to CF, ſo is the rectilineal figure ABC to KGH: But as

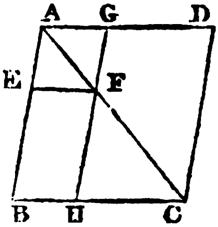


f 1. 6. BC to CF, ſo is [f] the parallelogram BE to the parallelogram
EF: Therefore, as the rectilineal figure ABC is to KGH, ſo is
g 11. 5. the parallelogram BE to the parallelogram EF [g]: And the rectilineal figure ABC is equal to the parallelogram BE; therefore
h 14. 5. the rectilineal figure KGH is equal [h] to the parallelogram EF:
But EF is equal to the figure D; wherefore alſo KGH is equal to D; and it is ſimilar to ABC. Therefore the rectilineal figure KGH has been deſcribed ſimilar to the figure ABC, and equal to D. Which was to be done.

PROP. XXVI. THEOR.

IF two ſimilar and ſimilarly placed parallelograms have a common angle; they are about the ſame diameter.

Let the parallelograms ABCD, AEFG be ſimilar and ſimilarly ſituated, and have the angle DAB common: They are about the ſame diameter.



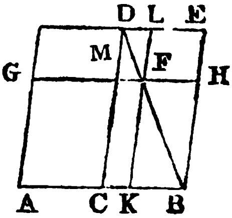
Join AF, FC, and produce GF to H: and becauſe the parallelograms BD, GE are ſimilar, DA is to AB, as [a] GA to AE: therefore the remainder DG is to the remainder BE, as GA to AE [b]: But DG is equal to HC [c], BE to HF, and AE to GF; therefore CH is to HF, as AG to GF: and the angle CHF is equal to AGF [d]; therefore the triangles CHF, AGF are equiangular [e]; and the angle CFH is therefore equal to AFG: and GFH is a ſtraight line; therefore AF and FC are in the ſame ſtraight line [f]. Wherefore, &c. Q. E. D.

a 1. Def. 6.
b Cor. 19. 5.
c 34. 1.
d 29. 1.
e 6. 6.
f 3. Cor. 15. 1.

PROP. XXVII. THEOR.

See N.

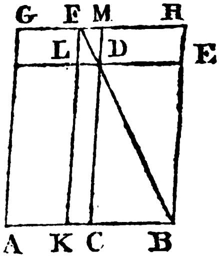
OF all equiangular parallelograms contained by ſegments of the ſame ſtraight line, and ſtraight lines which have to the remaining ſegments the ſame ratio, the greateſt is that which is deſcribed upon the half of the line.



Let AB be a ſtraight line divided into two equal parts in C; and upon AC let the parallelogram AD be deſcribed, and join BD: and let AK be any other part of AB; and draw [a] KF parallel to CD, and FG parallel to AB: and becauſe the angles DCB, FKB are equal, and DBC common to the triangles DCB, FKB, therefore DC is to CB, as [b] FK to KB: The equiangular parallelograms AD, AF are therefore contained by the ſegments AC, AK, and by DC, KF which have the ſame ratio to the remaining ſegments CB, KB: The parallelogram AD is greater than AF. Complete the parallelogram CE, and produce GF to meet BE in H, and let KF meet DE in L.

a 31. 1.
b 4. 6.

First, Let AK be greater than AC: and becauſe the comple-
ment CF is equal [c] to FE; the whole DH is greater than CF:
c 43. 1. But DH is equal [d] to DG, for HM is equal to MG, becauſe
d 36. 1. BC is equal to CA; therefore DG is
greater than CF: to each of theſe add
CG; then the whole AD is greater than
the whole AF.

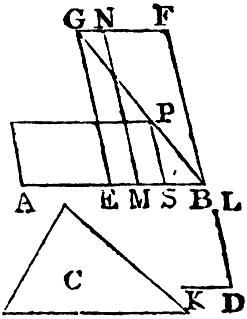


Next, Let AK be leſs than AF: and be-
c 43. 1. cauſe the complement DK is equal [c] to
d 36. 1. DH; and that DH is equal [d] to DG, for
HM is equal to MG, becauſe BC is equal
to CA; therefore DK is equal to DG;
DK is therefore greater than LG: to each
of theſe add AL; then the whole AD is
greater than the whole AF. Wherefore, &c. Q. E. D.

PROP. XXVIII. PROB.

See N. TO divide a given ſtraight line into two parts, ſo that the parallelogram having a given angle contained by one of the parts, and the ſtraight line which has a given ratio to the other part, ſhall be equal to a given rectilineal figure: But this figure muſt not be greater than the parallelogram upon half of the line, having the given angle contained by ſides in the given ratio.

Let AB be the given ſtraight line, and C the given rectilineal figure, and KDL the given angle contained by the ſides KD, DL, which have a given ratio to one another: and let C not be greater than the parallelogram upon the half of AB, having an angle equal to KDL contained by ſides which are to one another as KD to DL. It is required to divide AB into two parts, ſo that the parallelogram having an angle equal to KDL contained by one of theſe parts, and the ſtraight line, which is to the other part as LD to DK, ſhall be equal to C.



a 10. 1. Biſect [a] AB in E, and at B make [b]
b 23. 1. the angle EBF equal to KDL, and as
c 12. 6. KD is to DL, ſo make [c] EB to BF, and complete the parallelo-
gram EF: Then, if EF be equal to C, what was required is
done; for EF has the angle EBF equal to KDL contained by

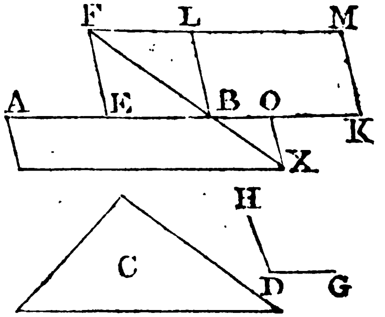
BE one of the parts of AB, and BF, which is to the other part AE, as LD to DK.

But, if EF be not equal to C, it is greater than it. To BF apply [d] the parallelogram MF equal to C, and having EBF for d Cor.45.1.
one of its angles: and find [e] ES a mean proportional between e 13. 6.
BE and EM; and join BG, and draw [f] SP parallel to GE or f 31. 1.
BF, meeting BG in P, and complete the parallelogram AP; it is equal to C. Becauſe EB is to ES, as ES to EM, by converſion [g], EB is to BS, as ES to SM: and as one of the antece- g E. 5.
dents to its conſequent, ſo are all the antecedents to all the conſequents [h]; therefore, as EB to BS, ſo is AS to BM: But as EB h 12. 5.
to BS, ſo [k] is EG or BF [l] to SP: therefore, as AS to BM, ſo k 4. 6.
[m] is FB to SP; that is, the ſides of the parallelograms AP, MF l 34. 1.
about their equal angles are reciprocally proportional; therefore m 11. 5.
the parallelogram AP is equal [n] to FM: but FM is equal to C; n 14. 6.
therefore AP is equal to C. Wherefore AB is cut in S, ſo that the parallelogram AP, having the angle ASP equal to KDL, contained by AS, and SP, which is to SB, as LD to DK, is equal to C. Which was to be done.

PROP. XXIX. PROB.

TO produce a given ſtraight line, ſo that the parallelogram having a given angle contained by the whole produced line, and the ſtraight line which has a given ratio to the produced part, ſhall be equal to a given rectilineal figure.

Let AB be the given ſtraight line, and C the given rectilineal figure, and GDH the given angle of which the ſides have a given ratio to one another: it is required to produce AB beyond B, ſo that the parallelogram having an angle equal to GDH, contained by the whole produced line, and the ſtraight line, which is to the produced part as HD to DG, ſhall be equal to C.



Biſect [a] AB in E, and a 10. 1.
produce it to K, and at B
make [b] the angle LBK equal b 23. 1.
to GDH, and as GD to
DH, ſo [c] make EB to BL, c 12. 6
and complete the parallelogram EL; and to BL apply [d] the parallelogram LK d Cor.45.1.
equal to C, having LBK for one of its angles; and between EB and EK find [e] a e 13. 6.
mean proportional EO; and

X join

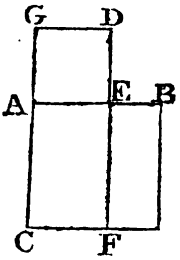
join FB, and draw [f] OX parallel to EF or BL, meeting FB
produced in X, and complete the parallelogram AX; it is equal
f 31. 1. to C.

g 17. 5. Becauſe KE is to EO, as EO to EB; by diviſion [g], KO is to
OE, as OB to BE or EA: and as one of the antecedents to its
h 12. 5. conſequent, ſo are all the antecedents to all the conſequents [h];
therefore, as BK is to AO, ſo is OB to BE: but as OB to BE,
l 2. 6. ſo [l] is OX to EF or BL; therefore [m] KB is to OA, as OX to
m 11. 5. BL; that is, the ſides of the parallelograms LK, AX, about
their equal angles LBK, AOX, are reciprocally proportional;
n 14. 6. therefore AX is equal to LK [n]: and LK is equal to C; therefore AX is equal to C. Wherefore, the ſtraight line AB is produced to O, ſo that the parallelogram AX having the angle AOX equal to GDH contained by the whole AO, and the ſide OX, which is to the produced part OB, as HD to DG, is equal to the given rectilineal figure C. Which was to be done.

PROP. XXX. PROB.

To cut a given ſtraight line in extreme and mean ratio.

Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.



a 46. 1. Upon AB deſcribe [a] the ſquare BC; and produce [b] CA to
b 29. 6. G, ſo that the rectangle contained by CG, and GD, equal to
GA, be equal to BC: And becauſe AG is equal to GD, the figure AD is a ſquare: and becauſe BC is equal to CD, by taking the common part CE from each, the remainder BF is equal to the remainder AD: and theſe figures are equiangular; therefore their ſides about the equal angles are reciprocally pro-
c 14. 6. portional [c]: wherefore, as FE to ED, ſo is
d 34. 1. AE to EB: But FE is equal to AC [d], that
is, to AB; and ED is equal to AE; therefore, as BA to AE, ſo is AE to EB: but AB is greater than
e 14. 5. AE; wherefore AE is greater than EB [e].

Therefore the ſtraight line AB is cut in extreme and mean
f 3. Def. 6. ratio in E [f]. Which was to be done.

Otherwiſe,

Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.

Divide AB in the point C, ſo that the rectangle contained by AB, BC be equal to the ſquare of AC [g].

Then, becauſe the rectangle AB, BC is equal to the ſquare of AC, as BA to AC, ſo is AC to CB [h]: Therefore AB is cut in extreme and mean ratio in C [f]. Which was to be done.

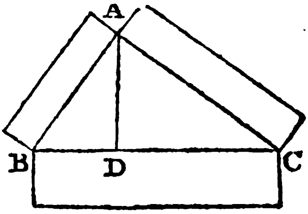
g 11. 2.
h 17. 6.
f 3. Def. 6.

PROP. XXXI. THEOR.

IN right angled triangles, the rectilineal figure deſcribed upon the ſide oppoſite to the right angle, is equal to the ſimilar, and ſimilarly deſcribed figures upon the ſides containing the right angle.

Let ABC be a right angled triangle, having the right angle BAC: The rectilineal figure deſcribed upon BC is equal to the ſimilar and ſimilarly deſcribed figures upon BA, AC.

Draw [a] the perpendicular AD; therefore, becauſe in the right angled triangle ABC, AD is drawn from the right angle at A perpendicular to the baſe BC, the triangles ABD, ADC are ſimilar to the whole triangle ABC, and to one another [b]: and becauſe the triangle ABC is ſimilar to ADB, as CB to BA, ſo is AB to BD [c]; and becauſe theſe three ſtraight lines are proportionals, as the firſt to the third, ſo is the figure upon the firſt to the ſimilar and ſimilarly deſcribed figure upon the ſecond [d]; therefore, as CB to BD, ſo is the figure upon CB to the ſimilar and ſimilarly deſcribed figure upon BA: and inverſely [e], as DB to BC, ſo is the figure upon BA to that upon BC: For the ſame reaſon, as DC to CB, ſo is the figure upon CA to that upon CB. Wherefore, as BD and DC together to BC, ſo are the figures upon BA, AC to that upon BC [f]: But BD and DC together are equal to BC; therefore the figure deſcribed on BC is equal [g] to the ſimilar and ſimilarly deſcribed figures on BA, AC. Wherefore, &c. Q. E. D.



a 12. 1.
b 8. 6.
c 4. 6.
d 2. Cor. 20. 6.
e B. 5.
f 24. 5.
g A. 5.

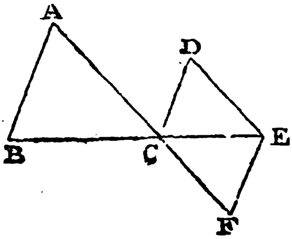
Book VI.

PROP. XXXII. THEOR.

IF two triangles which have two ſides of the one proportional to two ſides of the other, be joined at one angle, ſo as to have their homologous ſides parallel to one another; the remaining ſides ſhall be in a ſtraight line.

Let ABC, DCE be two triangles which have the two ſides BA, AC proportional to the two CD, DE, viz. BA to AC, as CD to DE; and let AB be parallel to DC, and AC to DE, BC and CE are in a ſtraight line.

a 31. 1. b 34. 1. Draw [a] EF parallel to AB or CD, and let it meet AC produced in F; therefore CDEF is a parallelogram [b], and CD is therefore equal [b] to EF, and DE to CF: But BA is to AC, as CD to DE; therefore BA is to AC, as EF to FC: and the alternate angles BAC, c 29. 1. CFE are equal [c]; therefore the triangles ABC, CFE are equiangular, and have the angle ACB equal to ECF: But ACF is a ſtraight line; therefore BC and CE are in the ſame ſtraight line. Wherefore, &c. Q. E. D.

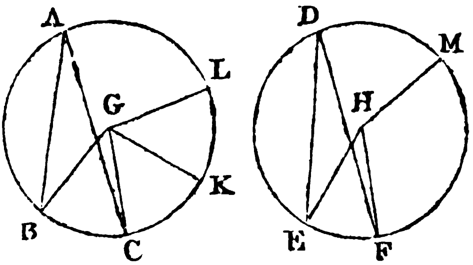


PROP. XXXIII. THEOR.

IN equal circles, angles, whether at the centres or circumferences, have the ſame ratio which the arches on which they ſtand have to one another: ſo alſo have the ſectors.

Let ABC, DEF be equal circles; and at their centres the angles BGC, EHF, and the angles BAC, EDF at their circumferences; as the arch BC to the arch EF, ſo is the angle BGC to the angle EHF, and the angle BAC to the angle EDF; and alſo the ſector BGC to the ſector EHF.

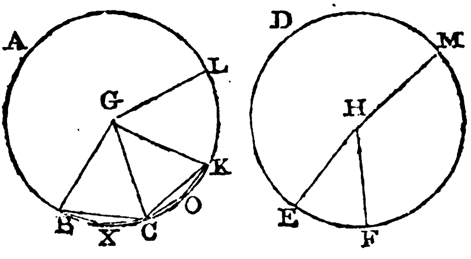
Take any number of arches CK, KL, each equal to BC, ſo that BL be greater than EF; and join GK, GL: And becauſe the arches BC, CK, KL are all equal, the angles BGC, CGK, a 27. 3. KGL are alſo all equal [a]; therefore the arch BL is the ſame multiple of the arch BC, that the angle BGL is of the angle BGC. Take, in like manner, the arch EM the leaſt multiple of the arch EF, that is greater than the arch BL, and join HM; and it may be proved as before, that the angle EHM is the ſame multiple of the angle EHF: conſequently the angle FHM contains EHF the ſame number of times that the arch FM contains EF: and if the arch BL be equal to FM, the angle BGL is alſo equal [a] to FHM: If not, the arch BL is greater than FM, but leſs than EM the next greater multiple of EF; and the angle BGL is greater than FHM, but leſs than EHM: therefore each of the arches BL, FM contains EF the ſame number of times, and each of the angles BGL, FHM contains EHF the ſame number of times: But the arch FM and the angle FHM contain EF and EHF equally; therefore the arch BL and angle BGL contain EF and EHF equally: and BL and BGL are any equimultiples of BC and BGC; as many times, therefore, as any multiple of BC contains EF, ſo many times does the ſame multiple of the angle BGC contain EHF; therefore, as the arch BC is to the arch EF, ſo [b] is the angle BGC to the angle EHF. But as the angle BGC to the angle EHF, ſo [c] is the angle BAC to EDF, for each is double of each [d]: Therefore, as the arch BC to the arch EF, ſo is the angle BGC to EHF, and the angle BAC to the angle EDF.



a 27. 3.
b 5. Def. 5.
c 15. 5.
d 20. 3.

Alſo, as the arch BC to EF, ſo is the ſector BGC to the ſector EHF.

Join BC, CK, and in the arches BC, CK take any points X, O, and join BX, XC, CO, OK: Then, becauſe in the triangles GBC, GCK the two ſides BG, GC are equal to the two CG, GK, and that they contain equal angles; the baſe BC is equal [e] to CK, and the triangle BGC to CGK: and becauſe the arch BC is equal to CK; if each of them be taken from the whole circumference, the remainder BAKC is equal to the remainder CBAK; therefore the angle BXC is equal [a] to the angle COK; and the ſegment



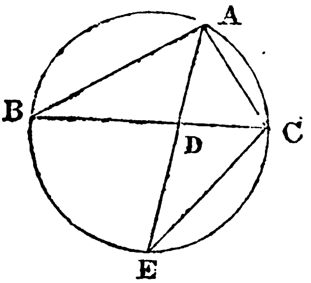
e 4. 1.
a 27. 3.

ment BXC is therefore ſimilar [f] to the ſegment COK: and they
are upon equal baſes BC, CK; therefore they are equal [g] to one
f 11.Def.3. another: and the triangle BGC is equal to the triangle CGK;
g 24. 3. therefore the whole ſector BGC is equal to the whole ſector
CGK. For the ſame reaſon, the ſector KGL is equal to each of the ſectors BGC, CGK: Therefore the ſector BGL is the ſame multiple of the ſector BGC, that the arch BL is of the arch BC. In the ſame manner, it may be proved, that the ſector EHM is the ſame multiple of the ſector EHF, that the arch EM is of the arch EF; and that, if the arch BL be equal to the arch FM, the ſector BGL is equal to the ſector FHM; and if BL be not equal to FM, it is greater than it, but leſs than EM; therefore the ſector BGL is greater than the ſector FHM, but leſs than the ſector EHM: Wherefore the arch BL and the ſector BGL contain the arch EF and ſector EHF the ſame number of times that the arch FM and ſector FHM contain them, that is, equally: as many times, therefore, as any multiple of the arch BC contains the arch EF, ſo many times does the ſame multiple of the ſector BGC contain the ſector
b 5.Def.5. EHF; therefore, as the arch BC to the arch EF, ſo [b] is the
ſector BGC to the ſector EHF. Wherefore, &c. Q. E. D.

PROP. B. THEOR.

IF an angle of a triangle be biſected by a ſtraight line, which alſo cuts the baſe; the rectangle contained by the ſides of the triangle is equal to the rectangle contained by the ſegments of the baſe, together with the ſquare of the ſtraight line biſecting the angle.

Let ABC be a triangle, and let the angle BAC be biſected by the ſtraight line AD; the rectangle BA, AC is equal to the rectangle BD, DC, together with the ſquare of AD.



a 5. 4. Deſcribe [a] the circle ACB about
the triangle, and produce AD to E,
and join EC: Then, becauſe the
angle BAD is equal to CAE, and
b 21. 3. the angle ABD to the angle AEC [b],
for they are in the ſame ſegment
ABEC; the triangles ABD, AEC
are equiangular: therefore, as BA
c 4. 6. to AD, ſo [c] is EA to AC; and
conſequently the rectangle BA, AC
is

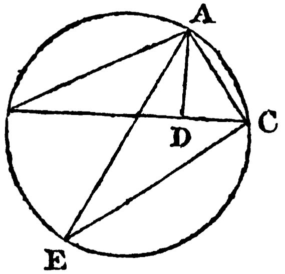
is equal [d] to the rectangle EA, AD, that is [e], to the rectangle ED, DA, together with the ſquare of AD: But the rectangle ED, DA is equal [f] to the rectangle BD, DC; therefore the rectangle BA, AC is equal to the rectangle BD, DC, together with the ſquare of AD. Wherefore, &c. Q. E. D.

d 16. 6.
e 3. 2.
f 35. 3.

PROP. C. THEOR.

If from any angle of a triangle, a ſtraight line be drawn perpendicular to the baſe; the rectangle contained by the ſides of the triangle, is equal to the rectangle contained by the perpendicular and the diameter of the circle deſcribed about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the baſe BC; the rectangle BA, AC is equal to the rectangle contained by AD and the diameter of the circle deſcribed about the triangle.



Deſcribe [a] the circle ACB about the triangle, and draw its diameter AE, and join EC: Becauſe the angle BDA is equal [b] to the angle ECA in a ſemicircle, and the angle ABD to the angle AEC in the ſame ſegment [c]; the triangles ABD, AEC are equiangular: Therefore, as [d] BA to AD, ſo is EA to AC; and conſequently the rectangle BA, AC is equal [e] to the rectangle EA, AD. If, therefore, &c. Q. E. D.

a 5. 4.
b 31. 3.
c 21. 3.
d 2. 6.
e 16. 6.

PROP. D. THEOR.

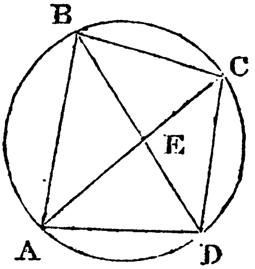
The rectangle contained by the diagonals of a quadrilateral inſcribed in a circle, is equal to both the rectangles contained by its oppoſite ſides.

Let ABCD be a quadrilateral inſcribed in a circle, and draw the diagonals AC, BD; the rectangle AC, BD is equal to the two rectangles AB, CD and AD, BC.

If the angle ABD be equal to DBC, let AC, BD meet in E: But, if they be not equal, let ABD be the greater; and make [a] the angle ABE equal to the angle DBC; therefore, if the angle

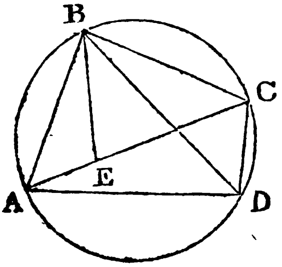
a 23. 1.

angle DBE be added to each of them, the angle ABD is equal to EBC.



Becauſe the angle ABD is equal to EBC, and the angle BDA to the angle BCE, for they are in the ſame ſegment BCDA[b]; the triangles ABD, BCE are equiangular: Wherefore[c], as BD to DA, ſo is BC to CE; and conſequently the rectangle BD, CE is equal[d] to the rectangle AD, BC. Again, becauſe the angle ABE is equal to the angle DBC, and the angle BAE to the angle[b] BDC; the triangle ABE is equiangular to BCD: as, therefore, BA to AE, ſo[c] is BD to DC; wherefore the rectangle BA, DC is equal to the rectangle BD, AE: But the rectangle BC, AD has been ſhown equal to the rectangle BD, CE; therefore the whole rectangle AC, BD[e] is equal to the rectangle AB, DC, together with the rectangle AD, BC. Wherefore, &c. Q. E. D.

b 21. 3.
c 4. 6.
d 16. 6.
e 1. 2.



Cor. If AD be equal to DC, or the angle ABD equal to DBC, the rectangle AC, BD is equal to the rectangles AD, BC and AD, AB; that is, to the rectangle contained by AD and the ſum of AB and BC. Conſequently AB, BC together are to BD, as AC to AD.

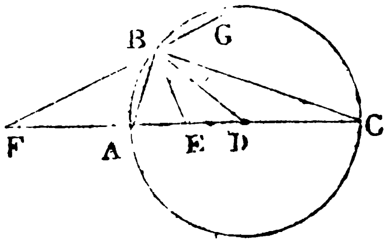
PROP. E. THEOR.

If two points be taken in the ſemidiameter of a circle, ſuch that the rectangle contained by the ſegments between them and the centre is equal to the ſquare of the ſemidiameter: ſtraight lines drawn from theſe points to any point of the circumference ſhall have the ſame ratio that the ſegments of the diameter between them and the circumference have to one another.

Let ABC be a circle, of which AC is the diameter, and D the centre; and let E, F be two points in AC, on the ſame ſide of the centre, ſo that the rectangle ED, DF is equal to the ſquare of AD; and draw EB, FB to any point B of the circumference: EB is to FB, as EA to AF.

Join AB, BD; and becauſe the rectangle ED, DF is equal to the ſquare of AD or DB, FD is to DB, as [a] DB to DE; that is, the ſides of the triangles FBD, EBD, about their common angle D, are proportionals; therefore the triangles are equiangular [b], and have the angle FBD equal to BED: But BED is equal [c] to the two EAB, ABE; therefore the two EAB, ABE are equal to FBD: of which EAB is equal [d] to ABD, beeauſe BD is equal to DA; therefore the remaining angle ABE is equal to the remaining angle ABF; that is, the angle FBE is biſected by BA: Wherefore, as FB to BE, ſo [e] is FA to AE. Therefore, &c. Q. E. D.

a 17. 6.
b 6. 6.
c 32. 1.
d 5. 1.
e 3. 6.



Cor. Hence, AB biſects the angle FBE. And if BC be joined, and FB produced to G: becauſe the angle ABC in a ſemicircle is a right angle [f], it is half the ſum of the angles FBE and EBG [g]; of which the angle ABE is the half of FBE; therefore the remaining angle EBC is the half of the remaining angle EBG: Therefore BC biſects the exterior angle EBG.

f 31. 3.
g 13. 1.

THE
ELEMENTS
OF
EUCLID.

BOOK XI.

DEFINITIONS.

I.

Book XI. A Solid is that which hath length, breadth, and thickneſs.

II.

That which bounds a ſolid is a ſuperficies.

III.

A ſtraight line is perpendicular, or at right angles to a plane, when it makes right angles with every ſtraight line meeting it in that plane.

IV.

A plane is perpendicular to a plane, when the ſtraight lines drawn in one of the planes perpendicularly to the common ſection of the two planes, are perpendicular to the other plane.

V.

* The inclination of a ſtraight line to a plane is the acute angle contained by that ſtraight line, and another drawn from the point in which the firſt line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the firſt line above the plane, meets the ſame plane.

VI.

See N. The inclination of a plane to a plane is the angle contained by two ſtraight lines drawn from any the ſame point of their common ſection at right angles to it, one upon one plane, and the other upon the other plane.

VII.

Two planes are ſaid to have the ſame, or a like inclination to one another, which two other planes have, when the ſaid angles of inclination are equal to one another.

VIII.

Parallel planes are ſuch as do not meet one another though produced.

IX.

Similar ſolid figures are ſuch as are contained by the ſame number of ſimilar planes having the ſame inclination to one another. See N.

X. Omitted. See N.

XI.

A ſolid angle is that which is made by the meeting of more than two plane angles, which are not in the ſame plane, in one point, the inclinations of all the planes being inwards. See N.

A.

A parallelopiped is a ſolid figure contained by ſix quadrilateral figures, whereof every oppoſite two are parallel.

XII.

A pyramid is a ſolid figure contained by planes that are conſtituted betwixt one plane and one point above it in which they meet.

XIII.

A priſm is a ſolid figure contained by plane figures of which two that are oppoſite are equal, ſimilar, and parallel to one another; and the others parallelograms.

XIV.

A ſphere is a ſolid figure deſcribed by the revolution of a ſemicircle about its diameter, which remains unmoved.

XV.

The axis of a ſphere is the fixed ſtraight line about which the ſemicircle revolves.

XVI.

The centre of a ſphere is the ſame with that of the ſemicircle.

XVII.

The diameter of a ſphere is any ſtraight line which paſſes through the centre, and is terminated both ways by the ſuperficies of the ſphere.

XVIII.

A cone is a ſolid figure deſcribed by the revolution of a right angled triangle about one of the ſides containing the right angle, which ſide remains fixed.

XIX.

The axis of a cone is the fixed ſtraight line about which the triangle revolves.

XX.

The baſe of a cone is the circle deſcribed by that ſide containing the right angle, which revolves.

XXI.

A cylinder is a ſolid figure deſcribed by the revolution of a right angled parallelogram about one of its ſides, which remains fixed.

XXII.

The axis of a cylinder is the fixed ſtraight line about which the parallelogram revolves.

XXIII.

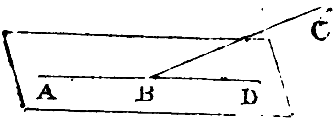
The baſes of a cylinder are the circles deſcribed by the two revolving oppoſite ſides of the parallelogram.

XXIV.

Similar cones and cylinders are thoſe which have their axes and the diameters of their baſes proportionals.

PROP. I. THEOR.

ONE part of a ſtraight line cannot be in a plane, and another part above it.

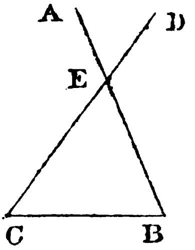


If it be poſſible, let AB, part of the ſtraight line ABC, be in the plane, and the part BC above it: And ſince the ſtraight line AB is in the plane, it can be produced in that plane: Let it be produced to D: And let any plane paſs through the ſtraight line AD, and be turned about it until it paſs through the point C; and becauſe the points B, C, are in this plane,
a 7.Def.1. the ſtraight line BC is in it [a]: Therefore there are two ſtraight
lines ABC, ABD in the ſame plane that have a common ſeg-
b 10.Ax.1. ment AB; which is impoſſible [b]. Therefore one part, &c.
Q. E. D.

PROP. II. THEOR.

TWO ſtraight lines which cut one another are in one plane, and three ſtraight lines which meet one another are in one plane.

Let two ſtraight lines AB, CD cut one another in E; AB, CD are one plane: And three ſtraight lines EC, CB, BE which meet one another, are in one plane.



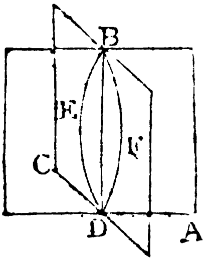
Let any plane paſs through the ſtraight line EB, and let the plane be turned about EB, produced, if neceſſary, until it paſs through the point C: Then, becauſe the points E, C are in this plane, the ſtraight line EC is in it [a]: For the ſame reaſon, the ſtraight line BC is in the ſame; and, by the hypotheſis, EB is in it: Therefore the three ſtraight lines EC, CB, BE are in one plane: But in the plane in which EC, EB are, in the ſame are [b] CD, AB: Therefore AB, CD are in one plane. Wherefore two ſtraight lines, &c. Q. E. D.

a 7 Def. 1.

b 1. 11.

PROP. III. THEOR.

IF two planes cut one another, their common ſection is a ſtraight line.



Let two planes AB, BC, cut one another, and let the line DB be their common ſection: DB is a ſtraight line: If it be not, from the point D to B, draw, in the plane AB, the ſtraight line DEB, and in the plane BC the ſtraight line DFB: Then two ſtraight lines DEB, DFB have the ſame extremities, and therefore include a ſpace betwixt them; which is impoſſible [a]: Therefore BD, the common ſection of the planes AB, BC, cannot but be a ſtraight line. Wherefore, if two planes, &c. Q. E. D.

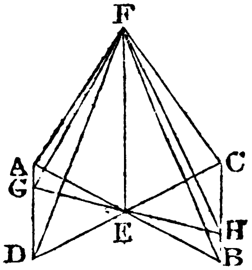
a 10. Ax. 1.

PROP. IV. THEOR.

IF a ſtraight line ſtand at right angles to each of two ſtraight lines in the point of their interſection, it ſhall alſo be at right angles to the plane which paſſes through them, that is, to the plane in which they are.

Let the ſtraight line EF ſtand at right angles to each of the ſtraight lines AB, CD in E, the point of their interſection: EF is alſo at right angles to the plane paſſing through AB, CD.

Take the ſtraight lines AE, EB, CE, ED all equal to one
another; and through E draw, in the plane in which are AB,
CD, any ſtraight line GEH; and join AD, CB; then, from any
point F in EF, draw FA, FG, FD, FC, FH, FB: And becauſe
the two ſtraight lines, AE, ED are equal to the two BE, EC,
a 15. 1. and that they contain equal angles [a] AED, BEC, the baſe AD
b 4. 1. is equal [b] to BC, and the angle DAE to EBC: And the angle
AEG is equal to BEH [a]; therefore the triangles AEG, BEH
have two angles equal to two, and the ſides AE, EB, adjacent
c 26. 1. to equal angles, are equal; the ſide GE is therefore [c] equal to
EH, and AG to BH: And becauſe AE is equal to EB, and
FE common, and at right angles to them, the baſe AF is equal [b]
to FB; for the ſame reaſon, CF is equal to FD: And becauſe
AD is equal to BC, and AF to FB, the two ſides FA, AD are
equal to FB, BC, each to each; and
the baſe DF was proved equal to FC;
d 8. 1. therefore the angle FAD is equal [d] to
FBC: Again, it was proved, that GA
is equal to BH, and alſo AF to FB;
FA, then, and AG, are equal to FB
and BH, and the angle FAG has been
proved equal to FBH; therefore the
baſe GF is equal [b] to FH: Again,
becauſe it was proved, that GE is
equal to EH, and EF is common; GE,
EF are equal to HE, EF; and the baſe GF is equal to FH;
therefore the angle GEF is equal [d] to HEF; and conſequently
e 10. Def. 1. each of theſe angles is a right [e] angle; therefore FE makes right
angles with GH. In like manner, it may be proved, that FE
makes right angles with every ſtraight line which meets it in the
plane paſſing through AB, CD. But a ſtraight line is at right
angles to a plane when it makes right angles with every ſtraight
f 3. Def. 11. line which meets it in that plane [f]: Therefore EF is at right
angles to the plane in which are AB, CD. Wherefore, if a
ſtraight line, &c. Q. E. D.



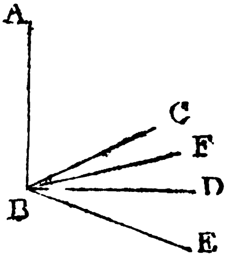
PROP. V. THEOR.

IF three ſtraight lines meet all in one point, and a ſtraight line ſtands at right angles to each of them in that point; theſe three ſtraight lines are in one and the ſame plane.

Let the ſtraight line AB ſtand at right angles to each of the ſtraight lines BC, BD, BE, in B the point where they meet; BC, BD, BE are in one and the ſame plane.

If not, let, if it be poſſible, BD and BE be in one plane, and BC be above it; and let a plane paſs through AB, BC, the common ſection of which, with the plane in which BD and BE are, ſhall be a ſtraight [a] line; let this be BF: Therefore the three ſtraight lines AB, BC, BF are all in one plane, viz. that which paſſes through AB, BC; and becauſe AB ſtands at right angles to each of the ſtraight lines BD, BE, it is alſo at right angles [b] to the plane paſſing through them; and therefore makes right angles [c] with every ſtraight line meeting it in that plane: But BF, which is in that plane, meets it; therefore the angle ABF is a right angle: but the angle ABC, by the hypotheſis, is alſo a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the ſame plane; which is impoſſible: Therefore the ſtraight line BC is not above the plane in which are BD and BE: Wherefore the three ſtraight lines BC, BD, BE are in one and the ſame plane. Therefore, if three ſtraight lines, &c. Q. E. D.

a 3. 11.
b 4. 11.
c 5. Def. 11.



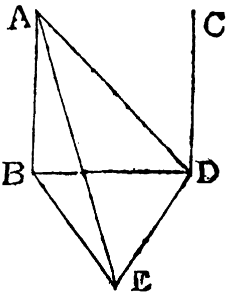
PROP. VI. THEOR.

IF two ſtraight lines be at right angles to the ſame plane, they ſhall be parallel to one another.

Let the ſtraight lines AB, CD be at right angles to the ſame plane; AB is parallel to CD.

Let them meet the plane in the points B, D, and draw the ſtraight line BD, to which draw DE at right angles, in the ſame plane; and make DE equal to AB, and join BE, AE, AD. Then, becauſe AB is perpendicular to the plane, it ſhall make right [a] angles with every ſtraight line which meets it, and is in that plane: But BD, BE, which are in that plane, do each of them meet AB; therefore each of the angles ABD, ABE is a right angle: For the ſame reaſon, each of the angles CDB, CDE is a right angle: And becauſe AB is equal to DE, and BD common, the two ſides AB, BD, are equal to the two ED, DB; and they contain right angles; therefore the baſe AD is equal [b] to the baſe BE: Again, becauſe AB is equal to DE, and BE to AD; AB, BE are equal to ED, DA; and, in the triangles ABE, EDA, the baſe

a 3. Def. 11.
b 4. 1.



baſe AE is common; therefore the angle ABE is equal [c] to the angle EDA: But ABE is a right angle; therefore EDA is c 8. 1. alſo a right angle, and ED perpendicular to DA: But it is alſo perpendicular to each of the two BD, DC: Wherefore ED is at right angles to each of the three ſtraight lines BD, DA, DC in the point in which they meet: Therefore theſe three ſtraight d 5. 11. lines are all in the ſame plane [d]: But AB is in the plane in which are BD, DA, becauſe any three ſtraight lines which meet e 2. 11. one another are in one plane [e]; therefore AB, BD, DC are in one plane: And each of the angles ABD, BDC is a right angle; f 28. 1. therefore AB is parallel [f] to CD. Wherefore, if two ſtraight lines, &c. Q. E. D.

PROP. VII. THEOR.

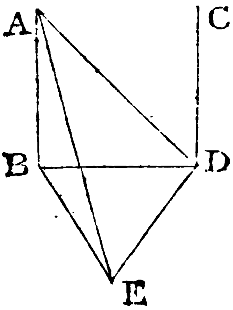
See N. IF two ſtraight lines be parallel, the ſtraight line drawn from any point in the one to any point in the other, is in the ſame plane with the parallels.

a 35. Def. 1. For the parallels are in the ſame plane [a]: and a ſtraight line b 7. Def. 1. joining two points in a plane lies wholly in that plane [b].

PROP. VIII. THEOR.

See N. IF two ſtraight lines be parallel, and one of them be at right angles to a plane; the other alſo ſhall be at right angles to the ſame plane.

Let AB, CD be two parallel ſtraight lines, and let one of them AB be at right angles to a plane; the other CD is at right angles to the ſame plane.

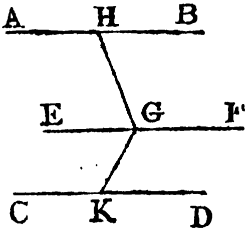


a 7. 11. Let AB, CD meet the plane in the points B, D, and join BD: Therefore [a] AB, CD, BD are in one plane. In the plane to which AB is at right angles, draw DE at right angles to BD, and make DE equal to AB, and join BE, AE, AD. And it may be demonſtrated, as in the ſixth propoſition, that the angles ABD, ADE are right angles; therefore DE is perpendicular to DA: But it is alſo perpendicular to BD; therefore ED b 4. 11. is perpendicular [b] to the plane which paſſes c 3. Def. 11. through BD, DA, and ſhall [c] make right angles with every ſtraight line meeting it in that plane: But DC is in the plane paſſing through BD, DA, becauſe all three are in the plane in which are the parallels AB, CD [a]: Where- a 7. 11. fore ED is at right angles to DC; and therefore CD is at right angles to DE: But CD is alſo at right angles to DB, for the interior angles ABD, BDC being equal to two right angles [d], d 29. 1. and ABD a right angle, CDB is alſo a right angle; CD then is at right angles to the two ſtraight lines DE, DB in the point of their interſection D; and therefore it is at right angles [b] to the b 4. 11. plane paſſing through DE, DB, which is the ſame plane to which AB is at right angles. Wherefore, &c. Q. E. D.

PROP. IX. THEOR.

TWO ſtraight lines which are each of them parallel to the ſame ſtraight line, and not in the ſame plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the ſame plane with it; AB ſhall be parallel to CD.



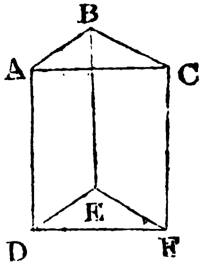
In EF take any point G, from which draw, in the plane paſſing through EF, AB, the ſtraight line GH at right angles to EF; and in the plane paſſing through EF, CD, draw GK at right angles to the ſame EF. And becauſe EF is perpendicular both to GH and GK, it is perpendicular [a] to a 4. 11. the plane HGK paſſing through them: And EF is parallel to AB; therefore AB is at right angles [b] to the plane b 8. 11. HGK. For the ſame reaſon, CD is likewiſe at right angles to the plane HGK. Therefore AB, CD are each of them at right angles to the plane HGK. But if two ſtraight lines be at right angles to the ſame plane, they are parallel [c]; therefore c 6. 11. AB is parallel to CD. Wherefore two ſtraight lines, &c. Q. E. D.

PROP. X. THEOR.

IF two ſtraight lines meeting one another be parallel to two others that meet one another, and are not in the ſame plane with the firſt two; the firſt two and the other two ſhall contain equal angles.

Book XI. Let the two ſtraight lines AB, BC which meet one another be parallel to the two ſtraight lines DE, EF that meet one another, and are not in the ſame plane with AB, BC; the angle ABC is equal to the angle DEF.

Take BA, BC, ED, EF all equal to one another; and join AD, CF, BE, AC, DF: Becauſe BA is equal and parallel to a 33. 1. ED, therefore AD is [a] both equal and parallel to BE. For the ſame reaſon, CF is equal and parallel to BE; therefore AD and CF are each of them equal and parallel to BE. But ſtraight lines that are parallel to the ſame ſtraight line, and not in the ſame b 9. 11. plane with it, are parallel [b] to one another; therefore AD is parallel to CF; and it is c 1. Ax. 1. equal [c] to it, and AC, DF join them towards the ſame parts; and therefore [a] AC is equal and parallel to DF. And becauſe AB, BC are equal to DE, EF, and the baſe AC to the baſe d 8. 1. DF; the angle ABC is equal [d] to the angle DEF. Therefore, if two ſtraight lines, &c. Q. E. D.

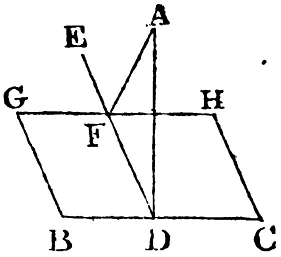


PROP. IX. PROB.

TO draw a ſtraight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH; it is required to draw from the point A a ſtraight line perpendicular to the plane BH.

In the plane draw any ſtraight line BC, and from the point a 12. 1. A draw [a] AD perpendicular to BC. If then AD be alſo perpendicular to the plane BH, the thing required is already done; but if it be not, from the point D b 11. 1. draw [b], in the plane BH, the ſtraight line DE at right angles to BC; and from the point A draw AF perpendi- c 31. 1. cular to DE; and through F draw [c] GH parallel to BC: And becauſe d 4. 11. BC is at right angles to ED and DA, BC is at right angles [d] to the plane paſſing through ED, DA. And GH is parallel to BC; but, if two e 8. 11. ſtraight lines be parallel, one of which is at right angles to a plane, the other ſhall be at right [e] angles f 3. Def. 11. to the ſame plane; wherefore GH is at right angles to the plane through ED, DA, and is perpendicular [f] to every ſtraight line meeting it in that plane. But AF, which is in the plane through ED,

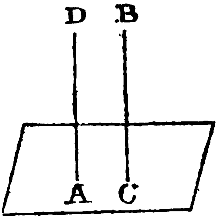


ED, DA, meets it; therefore GH is perpendicular to AF; and conſequently AF is perpendicular to GH: and AF is perpendicular to DE; therefore AF is perpendicular to each of the ſtraight lines GH, DE. But if a ſtraight line ſtands at right angles to each of two ſtraight lines in the point of their interſection, it ſhall alſo be at right angles to the plane paſſing through them. But the plane paſſing through ED, GH is the plane BH; therefore AF is perpendicular to the plane BH. Therefore, from the given point A, above the plane BH, the ſtraight line AF is drawn perpendicular to that plane. Which was to be done.

PROP. XII. PROB.

To erect a ſtraight line at right angles to a given plane, from a point given in the plane.

Let A be the point given in the plane; it is required to erect a ſtraight line from the point A at right angles to the plane.



From any point B above the plane draw [a] BC perpendicular to it; and from A draw [b] AD parallel to BC. Becauſe, therefore, AD, CB are two parallel ſtraight lines, and one of them BC is at right angles to the given plane, the other AD is alſo at right angles to it [c]. Therefore a ſtraight line has been erected at right angles to a given plane from a point given in it. Which was to be done.

a 11. 11.
b 31. 1.
c 8. 11.

PROP. XIII. THEOR.

THERE can be but one ſtraight line perpendicular to a given plane, drawn from the ſame point.

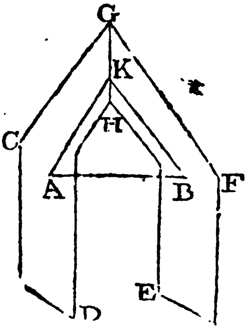
For, if there could be two, they would be parallel [a] to one another; which is abſurd. a 6. 11.

PROP. XIV. THEOR.

Planes to which the ſame ſtraight line is perpendicular, are parallel to one another.

Let the ſtraight line AB be perpendicular to each of the planes CD, EF; theſe planes are parallel to one another.

If not, they ſhall meet one another when produced; let
them meet; their common ſection ſhall be a ſtraight line GH,
in which take any point K, and join
AK, BK: Then, becauſe AB is perpendicular
to the plane EF, it is perpendicular
a 3.Def.11. [a] to the ſtraight line BK
which is in that plane; therefore
ABK is a right angle. For the ſame
reaſon, BAK is a right angle; wherefore
the two angles ABK, BAK of
the triangle ABK are equal to two
b 17. 1. right angles; which is impoſſible [b]:
Therefore the planes CD, EF, though
produced, do not meet one another;
c 8.Def.11. that is, they are parallel [c]. Therefore planes, &c. Q. E. D.

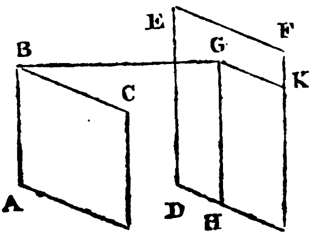


PROP. XV. THEOR.

IF two ſtraight lines meeting one another, be parallel to two ſtraight lines which meet one another, but are not in the ſame plane with the firſt two; the plane which paſſes through theſe is parallel to the plane paſſing through the others.

Let AB, BC, two ſtraight lines meeting one another, be parallel to DE, EF that meet one another, but are not in the ſame plane with AB, BC: The planes through AB, BC, and DE, EF ſhall not meet, though produced.

a 11. 11. From the point B draw BG perpendicular [a] to the plane
which paſſes through DE, EF, and let it meet that plane in G;
b 31. 1. and through G draw GH parallel [b]
to ED, and GK parallel to EF:
And becauſe BG is perpendicular
to the plane through DE, EF, it
ſhall make right angles with every
ſtraight line meeting it in that
c 3.Def.11. plane [c]: But the ſtraight lines GH,
GK in that plane meet it; therefore
each of the angles BGH, BGK
is a right angle: And becauſe BA
d 9. 11. is parallel [d] to GH, for each of them is parallel to DE, and
they are not both in the ſame plane with it, the angles GBA,
e 29. 1. BGH are together equal [e] to two right angles: And BGH is a
right angle; therefore alſo GBA is a right angle, and GB perpendicular



pendicular to BA: For the ſame reaſon, GB is perpendicular to BC: Since, therefore, the ſtraight line GB ſtands at right angles to the two ſtraight lines BA, BC, that cut one another in B; GB is perpendicular [f] to the plane through BA, BC: And it is perpendicular to the plane through DE, EF; therefore BG is perpendicular to each of the planes through AB, BC, and DE, EF: But planes to which the ſame ſtraight line is perpendicular, are parallel [g] to one another; therefore the plane through AB, BC is parallel to the plane through DE, EF. Wherefore, if two ſtraight lines, &c. Q. E. D.

f 4. 11.
g 14. 11.

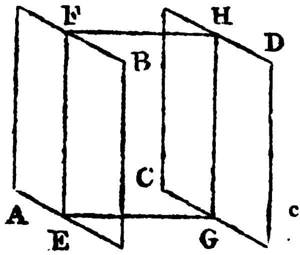
PROP. XVI. THEOR.

See N.

If two parallel planes be cut by another plane, their common ſections with it are parallels.

Let the parallel planes AB, CD be cut by the plane EFHG, and let their common ſections with it be EF, GH: EF is parallel to GH.

Becauſe the planes AB, CD are parallel, they do not meet [a], though produced: and the ſtraight lines EF, GH are wholly in theſe planes [b]; therefore EF, GH do not meet, though produced either way: And they are in the ſame plane EFGH: But ſtraight lines which are in the ſame plane, and do not meet though produced either way, are parallel [c]; therefore EF is parallel to GH. Wherefore, &c. Q. E. D.



a 8. Def. 11.
b 1. 11.
c 35. Def. 1.

PROP. XVII. THEOR.

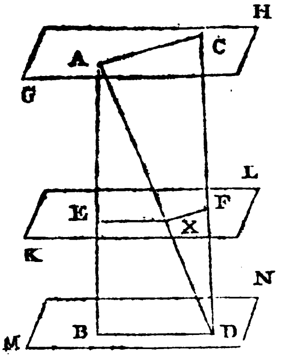
If two ſtraight lines be cut by parallel planes, they ſhall be cut in the ſame ratio.

Let the ſtraight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D: As AE is to EB, ſo is CF to FD.

Join AC, BD, AD, and let AD meet the plane KL in the point X; and join EX, XF: Becauſe the two parallel planes KL, MN are cut by the plane EBDX, the common ſections EX, BD, are parallel [a]. For the ſame reaſon, becauſe the two

a 16. 11.

parallel planes GH, KL are cut by the plane AXFC, the common ſections AC, XF are parallel: And becauſe EX is parallel to BD, a ſide of the triangle ABD, b 2. 6. as AE to EB, ſo is [b] AX to XD. Again, becauſe XF is parallel to AC, a ſide of the triangle ADC, as AX to XD, ſo is CF to FD: And it was proved, that AX is to XD, as c 11. 5. AE to EB; therefore [c], as AE to EB, ſo is CF to FD. Wherefore, if two ſtraight lines, &c. Q. E. D.

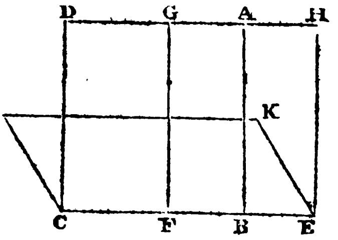


PROP. XVIII. THEOR.

IF a ſtraight line be at right angles to a plane, every plane which paſſes through it ſhall be at right angles to that plane.

Let the ſtraight line AB be at right angles to a plane CK; every plane which paſſes through AB ſhall be at right angles to the plane CK.

Let any plane DE paſs through AB, and let CE be the common ſection of the planes DE, CK; take any point F in CE, from which draw FG in the plane DE at right angles to CE: And becauſe AB is perpendicular to the plane CK, therefore it is alſo perpendicular to every ſtraight line in that plane meet- a 3.Def.11. ing it [a]; and conſequently it is perpendicular to CE; wherefore ABF is a right angle: but GFB is likewiſe a right angle; b 28. 1. therefore AB is parallel [b] to FG. And AB is at right angles to the plane CK; therefore FG is alſo at right angles to the ſame c 8. 11. plane [c]; that is, any ſtraight in the plane DE, at right angles to CE, the common ſection of the planes, is at right angles to the other plane CK: But one plane is at right angles to another plane, when the ſtraight lines drawn in one of the planes, at right angles to their common ſection, are alſo at right angles to d 4.Def.11. the other plane [d]; therefore the plane DE is at right angles to the plane CK. In like manner, it may be proved, that all the planes



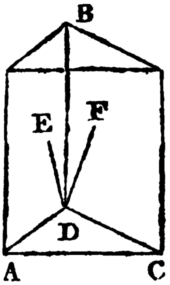
planes which paſs through AB are at right angles to the plane CK. Therefore, if a ſtraight line, &c. Q. E. D.

PROP. XIX. THEOR.

IF two planes cutting one another be each of them perpendicular to a third plane; their common ſection ſhall be perpendicular to the ſame plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common ſection of the firſt two; BD is perpendicular to the third plane.

If it be not, from the point D draw, in the plane AB, the ſtraight line DE at right angles to AD, the common ſection of the plane AB, with the third plane; and in the plane BC draw DF at right angles to CD, the common ſection of the plane BC, with the third plane. And becauſe the plane AB is perpendicular to the third plane, and DE is drawn in the plane AB at right angles to AD their common ſection, DE is perpendicular to the third plane [a]. In the ſame manner, it may be proved, that DF is perpendicular to the third plane. Wherefore, from the point D two ſtraight lines ſtand at right angles to the third plane, upon the ſame ſide of it; which is impoſſible [b]: Therefore, from the point D there cannot be any ſtraight line at right angles to the third plane, except BD, the common ſection of the planes AB, BC. BD, therefore, is perpendicular to the third plane. Wherefore, if two planes, &c. Q. E. D.



a 4. Def. 11.

b 13. 11.

PROP. XX. THEOR.

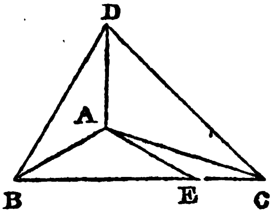
IF a ſolid angle be contained by three plane angles, any two of them are greater than the third.

Let the ſolid angle at A be contained by the three plane angles BAC, CAD, DAB. Any two of them are greater than the third.

If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third. But if they are not, let BAC be that angle which is not leſs than either of the other two, and is greater than one of them DAB; and at the point A in the ſtraight line AB, make, in the plane which paſſes through BA, AC, the angle BAE equal [a] to the angle

a 23. 1.

angle DAB; and make AE equal to AD, and through E draw BEC, cutting AB, AC in the points B, C, and join DB, DC. And becauſe DA is equal to AE, and AB is common, the two DA, AB are equal to the two EA, AB, and the angle DAB is equal to the angle EAB;
b 4. 1. therefore the baſe DB is equal [b] to the baſe BE. And becauſe BD,
c 20. 1. DC are greater [c] than CB, and one of them BD has been proved equal to BE a part of CB; therefore the other DC is greater than the remaining part EC. And becauſe DA is equal to AE, and AC common, but the baſe DC greater than the baſe EC; therefore
d 25. 1. the angle DAC is greater [d] than the angle EAC; and, by the conſtruction, the angle DAB is equal to the angle BAE; wherefore the angles DAB, DAC are together greater than BAE, EAC, that is, than the angle BAC. But BAC is not leſs than either of the angles DAB, DAC; therefore BAC, with either of them, is greater than the other. Wherefore, if a ſolid angle, &c. Q. E. D.

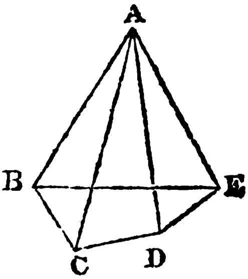


PROP. XXI. THEOR.

See N. EVERY ſolid angle is contained by plane angles, which together are leſs than four right angles.

Let the ſolid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAB. Theſe together are leſs than four right angles.

Let the planes in which the angles are be cut by a plane, and let their common ſections with it be BC, CD, DE, EB: Then, becauſe the ſolid angle at B is contained by the three plane angles CBA, ABE, EBC, any two of them are greater than the third; therefore the angles CBA, ABE are greater than the angle EBC: For the ſame reaſon, the two plane angles BCA, ACD are greater than BCD, and the two CDA, ADE greater than CDE; and DEA, AEB greater than DEB: Therefore all the angles at the baſes of the triangles ABC, ACD, ADE, ABE, which have their common vertex at A, are together greater than all the angles of the rectilineal figure BCDE: and becauſe the three angles of a triangle are equal to two right angles, all the angles of the triangles ABC, ACD, ADE, AEB are equal to twice as many right angles as there are triangles; that is, as there are ſides of the figure BCDE; and all the angles of the figure BCDE, together with four right angles, are alſo equal to twice as many right angles as there are ſides of the figure; therefore all the angles of the triangles ABC, ACD, ADE, AEB are equal to all the angles of the figure, together with four right angles: But it has been proved, that all the angles at the baſes of the triangles are greater than all the angles of the figure; wherefore the remaining angles at the vertex A of the triangles, are leſs than four right angles. Wherefore every ſolid angle, &c. Q. E. D.

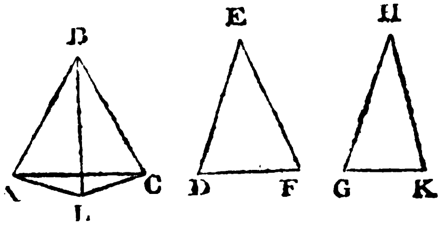


PROP. XXII. THEOR.

IF three iſoſceles triangles have their ſides equal, and any two of the vertical angles greater than the third; a triangle may be made of their baſes.

Let ABC, DEF, GHK be three iſoſceles triangles, having their ſides AB, BC, DE, EF, GH, HK all equal, and any two of the angles at B, E, H greater than the third; any two of the baſes AC, DF, GK are alſo greater than the third.

If two of the angles at B, E, H be equal, and not leſs than the third, it is evident that two of the baſes are equal [a] and not a 4. 1.
leſs [b] than the third; and therefore any two of the baſes are b 24. 1.
greater than the third. In any other caſe, let the angle ABC be greater than either of the other angles at E, H, therefore AC is greater [b] than either DF or GK. At B, in AB, make [c] the angle c 23. 1.
ABL equal to DEF, and BL equal to one of the equal ſides, and join AL, LC: and becauſe the two ſides AB, BL are equal to the two DE, EF, and they contain equal angles; therefore the baſe AL is equal [d] to DF: and becauſe the angles at E, H are greater d 24. 1.
than ABC, and that E is equal to ABL, the remaining angle at H is greater than the remaining angle CBL: and becauſe the two ſides LB, BC are equal to the two GH, HK, but the angle LBC leſs than GHK; therefore the baſe LC is leſs [b] than GK; and it was proved, that AL is equal to DF; therefore DF, GK are greater than AL, LC: but AL, LC are greater [e] than AC; e 20. 1.



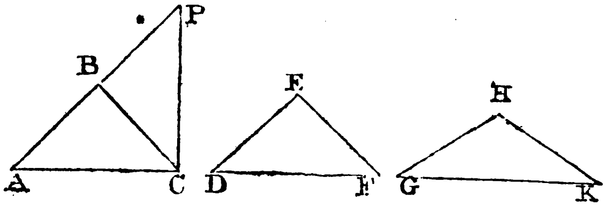
therefore DF, GK are greater than AC: and it is evident, that AC with either of the two DF, GK are greater than the other;
f 22. 1. therefore a triangle may be made [f], the ſides of which ſhall be equal to AC, DF, GK. Q. E. D.

PROP. XXIII. PROB.

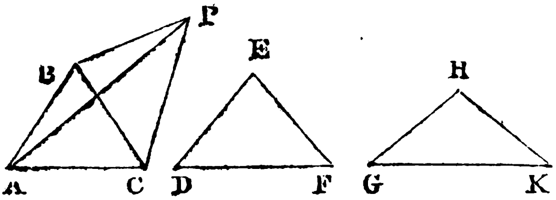
See N. TO make a ſolid angle which ſhall be contained by three given plane angles, any two of them being greater than the third, and all three together leſs than four right angles.

Let the three given plane angles be ABC, DEF, GHK, any two of which are greater than the third, and all of them together leſs than four right angles. It is required to make a ſolid angle contained by three plane angles equal to ABC, DEF, GHK, each to each.

From the ſides of the angles cut off AB, BC, DE, EF, GH, HK all equal to one another; and join AC, DF, GK: and let

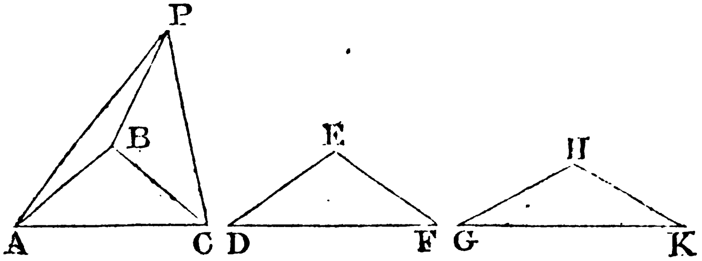


the angle GHK not be leſs than either of the other two; there-
a 4. or fore GK is not leſs [a] than either AC or DF. At the point B
24. 1. make [b] the angle CBP equal to DEF, and make BP equal to
b 23. 1. one of the equal ſides, and join CP: and becauſe the two ſides CB, BP are equal to the two DE, EF, and the angle CBP is
c 4. 1. equal to the angle DEF; the baſe CP is equal [c] to DF. If AB



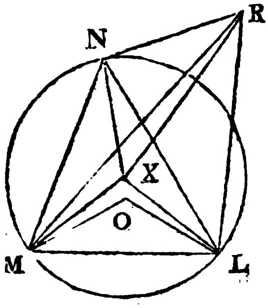
and BP be in the ſame ſtraight line; AP is equal to GH, HK to-
d 20. 1. gether; and GH, HK are greater [d] than GK; therefore AP is greater

greater than GK. But, if AB and BP be not in a ſtraight line, join AP: and if the point B be without the triangle ACP; becauſe the two angles ABC, DEF are greater than GHK; and CBP is equal to DEF; therefore the two ABC, CBP, that is, the angle ABP is greater than GHK: and the two ſides AB, BP are equal to the two GH, HK; therefore the baſe AP is greater[e] e 24. 1. than the baſe GK. But if the point B be within the triangle ACP, the three angles ABC, CBP, ABP are equal[f] to four f 2. Cor. right angles: and the three angles ABC, CBP, GHK are leſs 15. 1. than four right angles; therefore the angle ABP is greater than GHK: and the ſides AB, BP are equal to GH, HK; therefore the baſe AP is greater[e] than GK. Wherefore in every caſe AP is greater than GK.



And becauſe every two of the angles ABC, DEF, GHK are greater than the third, a triangle may be made of the three baſes AC, DF, GK: let this be the triangle LMN[g], ſo that LM be g 22. 11. equal to AC, MN to DF or CP, and LN to GK: and becauſe the two ſides AC, CP are equal to the two LM, MN, but the baſe AP greater than the baſe LN; therefore the angle ACP is greater[h] than the angle LMN. About the triangle LMN de- h 25. 1. ſcribe[k] a circle, and find[l] its centre X, and join XL, XM, XN: k 5. 4. AB is greater than XL: For if AB be equal to XL, BC is equal l 1. 3. to XM: and becauſe BC, CA are equal to XM, ML, and AB equal to XL; the angle ACB is equal to LMX[m]: For the ſame m 8. 1. reaſon, the angle BCP is equal to XMN; therefore the whole angle ACP is equal to the whole LMN: and it was ſhown to be greater than it; which is impoſſible; therefore AB is not equal to LX: Neither is it leſs, for, if it be, make the triangle LOM on the ſame ſide of LM with the triangle LXM, ſo that LO, OM be equal to AB, BC; therefore LO, OM are leſs than LX, XM; and therefore the triangle LOM falls within[n] the triangle n 21. 1. LXM. Conſequently the angle LMO is leſs than LMX: but LMO is equal[o] to ACB; therefore ACB is leſs than LMX. o 8. 1. For the ſame reaſon, the angle BCP is leſs than XMN; therefore the whole angle ACP is leſs than the whole LMN; and it

Book XI. is alſo greater than it; which is impoſſible: Therefore AB is not leſs than LX: and it has been proved, that it is not equal to it; therefore AB is greater than LX.



p 12. 11. From the point X erect [p] XR at
right angles to the plane of the circle
LMN: and find a ſquare equal to the
exceſs of the ſquare of AB above the
ſquare of LX, and make RX equal to
its ſide, and join RL, RM, RN. Becauſe RX is perpendicular to the plane
q 3. Def. 11. LMN, it is perpendicular [q] to each of
the ſtraight lines LX, MX, NX.
And becauſe LX is equal to MX,
and RX common, and at right angles
o 4. 1. to each of them, the baſe RL is equal [o] to RM.

For the ſame reaſon, RN is equal to each of the two RL, RM. And becauſe the ſquare of RX is equal to the exceſs of the ſquare of AB above that of LX; the ſquare of AB is equal to
r 47. 1. the ſquares of LX, XR: But the ſquare of LR is equal [r] to the ſame ſquares, becauſe LXR is a right angle; therefore the ſquare of AB is equal to the ſquare of LR, and the ſtraight line AB to LR: But each of the ſtraight lines BC, DE, DF, GH, HK is equal to AB, and each of the two RM, RN is equal to RL; therefore each of the former is equal to each of the latter: and becauſe RL, RM, are equal to AB, BC, and the baſe LM
m 8. 1. to the baſe AC; the angle LRM is equal [m] to the angle ABC.

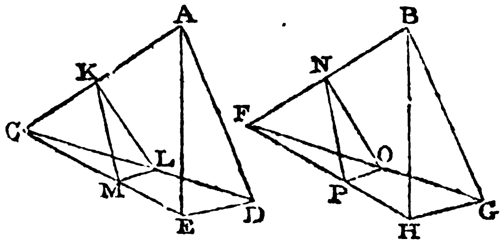
For the ſame reaſon, the angle MRN is equal to DEF, and NRL to GHK. Therefore there is made a ſolid angle at R, which is contained by three plane angles LRM, MRN, NRL, which are equal to the three given plane angles ABC, DEF, GHK, each to each. Which was to be done.

PROP. A. THEOR.

See N. IF each of two ſolid angles be contained by three plane angles equal to one another, each to each; the planes in which the equal angles are, have the ſame inclination to one another.

Let the ſolid angle at A be contained by the three plane angles CAD, CAE, EAD; and the ſolid angle at B by the three plane angles FBG, FBH, GBH; of which the angle CAD is equal to the angle FBG, and CAE to FBH, and EAD to HBG: The planes in which the equal angles are, have the ſame inclination to one another.

Make the ſtraight lines AC, AD, AE; BF, BG, BH all equal to one another, and join CD, CE, ED; FG, FH, HG: and becauſe the triangles are all iſoſceles, each of their angles at the points C, D, E, F, G, H, is leſs [a] than a right angle. In AC take any point K, and from it draw [b] KL in the plane CAD, and KM in the plane CAE, each of them at right angles to AC; therefore the angle LKM is the inclination [c] of the plane CAD to the plane CAE. In FB take FN equal to CK; and from N draw [b] NO in the plane FBG, and NP in the plane FBH, each of them at right angles to FB; therefore the angle ONP is the inclination [c] of the plane FBG to the plane FBH: join LM, OP: and becauſe in the triangles CAD, FBG, CA, AD are equal to FB, BG; and the angle CAD equal to FBG; the baſe CD is equal [d] to FG, and the other angles to the other angles. For the ſame reaſon, CE is equal to FH, and ED to GH: and becauſe DC, CE are equal to GF, FH, as alſo the baſe DE to the baſe GH; the angle DCE is equal [e] to GFH: and becauſe in the triangles CKL, FNO, the angles KCL, NFO are equal, as alſo the right angles CKL, FNO; and that the ſides CK, FN, adjacent to the equal angles, are alſo equal; the ſide KL is equal [f] to NO, and CL to FO: For the ſame reaſon, in the triangles CKM, FNP, KM is equal to NP, and MC to FP: and in the triangles MCL, PFO, the ſides MC, CL are equal to PF, FO, and the angle MCL to the angle PFO; therefore the baſe LM is equal [d] to OP. Laſtly, in the triangles KML, NPO, the ſides MK, KL are equal to PN, NO, and the baſe LM to the baſe OP; therefore the angle LKM is equal [e] to the angle ONP; that is, the inclination of the planes CAD, CAE is the ſame with the inclination of the planes FBG, FBH to one another. In the ſame manner, it may be demonſtrated, that the other planes in which the equal angles are, have the ſame inclination to one another. Therefore, &c. Q. E. D.



a 17. 1.
b 11. 1.
c 6. Def. 11.
d 4. 1.
e 8. 1.
f 26. 1.

Cor. Likewiſe, if two ſolid angles be each contained by three plane angles, and have two angles of the one equal to two angles of the other, and theſe angles be in planes which have the ſame inclination to one another; the third angle of the one is equal to the third angle of the other.

Let CAD, CAE two of the three plane angles which contain the ſolid angle at A, be equal to FBG, FBH two of the plane angles which contain the ſolid angle at B; and let the inclination of the plane CAD to the plane CAE, be the ſame with that of the plane FBG to the plane FBH; the third plane angle EAD is equal to the third plane angle GBH.

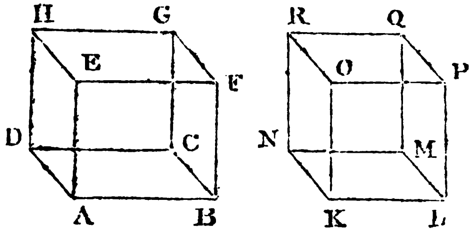
For the ſame conſtruction being made as in the propoſition, it may be proved as before, that the ſtraight lines DC, CE, KL, LC, KM, MC, are equal to GF, FH, NO, OF, NP, PF, each to each. And becauſe LK, KM are equal to ON, NP, and the angle d 4. 1. LKM to the angle ONP; the baſe LM is equal [d] to OP: and becauſe LC, CM are equal to OF, FP, and the baſe LM to OP; c 8. 1. therefore the angle LCM is equal [c] to OFP: and becauſe DC, CE are equal to GF, FH, and the angle DCE to GFH; the baſe DE is equal [d] to GH: and becauſe the two ſides DA, AE are equal to GB, BH, and the baſe DE alſo equal to the baſe GH; therefore the angle DAE is equal [c] to the angle GBH.

PROP. B. THEOR.

See N. SOLID figures contained by the ſame number of equal and ſimilar planes, alike ſituated, and having the ſame inclination to one another, are equal to one another.

Let AG, KQ be two ſolid figures contained by the ſame number of ſimilar and equal planes, having the ſame inclination to one another; and let theſe ſimilar planes have the ſame ſituation and order in each of the ſolids, viz. let the plane AC be ſimilar and equal to the plane KM; the plane AF to KP; BG to LQ; GD to QN; DE to NO; and FH to PR: The ſolid AG is equal to the ſolid KQ.

For, if the plane AC be applied to the plane KM, ſo that the ſide AB may coincide with the homologous ſide KL, and A with K; the point B coincides with L, becauſe AB is equal to KL: and the plane figure AC coincides with the figure KM, becauſe they are equal and ſimilar; therefore the ſtraight lines AD, DC, CB coincide with KN, NM, ML; and the point C with M, and D with N: and the planes AF, KP are, by the hypotheſis, on the ſame ſide of the planes AC, KM, and have the ſame inclination to them;



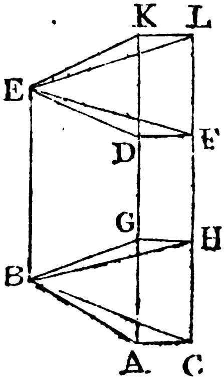
them; therefore the plane AF coincides with the plane KP, and the points E, F with O, P, becauſe the figures AF, KP are ſimilar and equal. In the ſame manner, it may be proved, that the other plane figures of the ſolid AG coincide with the plane figures ſimilar to them of the ſolid KP. Wherefore the ſolid AG coincides every way with the ſolid KP, and is therefore equal [a] to it. Wherefore, &c. Q. E. D. a 8. Ax. 1.

PROP. C. THEOR.

IF two triangular priſms have one of the ſides of their parallelograms common to both, and the ſides oppoſite to it in the ſame ſtraight lines; the priſms are equal to one another. See N.

Let ABC-DEF and BGH-EKL be two priſms, of which the baſes are the triangles ABC, GBH, and the oppoſite triangles EDF, KEL; and let the two parallelograms AE, EC of the firſt, and the two KB, BL of the other, have the ſide BE common to them, and their oppoſite ſides in the ſtraight lines AK, CL; the priſm ABC-DEF is equal to the priſm GBH-KEL.

Becauſe AE, EG are parallelograms; AD, GK are each equal to BE [a], and to one another: and DG is common; therefore a 34. 1. AG is equal to DK. For the ſame reaſon, CH is equal to FL: and AC is equal and parallel to DF [a]; and GH to KL; therefore the figure AH is equal and ſimilar to DL. For the ſame reaſon, ABG is equal and ſimilar to EDK; and CBH to FEL: and the triangles ABC, GBH are equal and ſimilar to DEF, KEL: and becauſe the plane angles BAC, BAG, GAC are equal to the angles EDF, EDK, KDF; the planes about the ſolid angles at A, D have the ſame inclination to one another [b]. b A. 11. For the ſame reaſon, the other planes have the ſame inclination to one another: and all theſe planes are on the ſame ſide of the common plane AL, and are in the ſame order. Therefore the ſolid ACHG-B is equal to the ſolid DFLK-E [c]: Take theſe equals from the whole ſolid c B. 11. ABC-EKL; and the remaining priſm GBH-KEL is equal to the remaining priſm ABC-DEF. Wherefore, &c. Q. E. D.

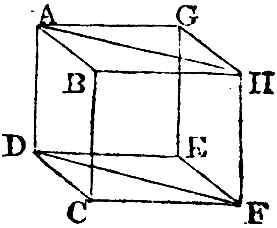


PROP. XXIV. THEOR.

IF a ſolid be contained by ſix planes, two and two of which are parallel; the oppoſite planes are ſimilar and equal parallelograms.

Let the ſolid CDGH be contained by the parallel planes AC, GF; BG, CE; FB, AE; its oppoſite planes are ſimilar and equal parallelograms.

Becauſe the parallel planes BG, CE are cut by the plane AC, a 16. 11. their common ſections AB, CD are parallels [a]. Again, becauſe the parallel planes BF, AE are cut by the plane AC, their common ſections AD, BC are parallel [a]; therefore AC is a parallelogram. In the ſame manner, it may be proved, that each of the figures CE, FG, GB, BF, AE is a parallelogram: Join AH, DF; and becauſe AB is parallel to CD, and BH to CF, and AB, BH are not in the ſame plane with DC, CF; therefore the b 10. 11. angle ABH is equal [b] to the angle DCF: And becauſe AB, BH are equal to DC, CF, and the angle ABH to DCF; therefore c 4. 1. the baſe AH is equal [c] to DF, and the triangle ABH to DCF: d 34. 1. and the parallelogram BG is double [d] of the triangle ABH, and CE of DCF; therefore the parallelogram BG is equal and ſimilar to CE. In the ſame manner, it may be proved, that the parallelogram AC is equal and ſimilar to GF, and AE to BF. Therefore, if a ſolid, &c. Q. E. D.

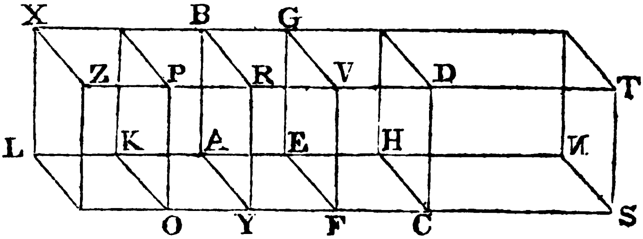


PROP. XXV. THEOR.

IF a ſolid parallelopiped be cut by a plane parallel to two of its oppoſite planes; it divides the whole into two ſolids, the baſe of one of which ſhall be to the baſe of the other, as the one ſolid is to the other.

Let the ſolid parallelopiped ABCD be cut by the plane EV, which is parallel to the oppoſite planes AR, HD, and divides the whole into the two ſolids ABFV, EGCD; as the baſe AEFY of the firſt is to the baſe EHCF of the other, ſo is the ſolid ABFV to the ſolid EGCD.

Produce AH both ways, and take any number of ſtraight
lines AK, KL each equal to EA, and complete the parallelograms
LO, KY, and the ſolids LP, KR: Then, becauſe the
ſtraight lines LK, KA, AE are all equal, as alſo KO, AY, EF
are equal [a] and parallel; the parallelograms LO, KY, AF are a 34. 1.
equal and ſimilar [b]: For the ſame reaſon, the parallelograms b {4. 1. & 34. 1.
KX, KB, AG are equal and ſimilar; as alſo the parallelograms
LZ, KP, AR: Therefore three planes of the ſolid LP, are equal
and ſimilar to three planes of the ſolid KR, as alſo to three planes
of the ſolid AV: But the three planes oppoſite to theſe three
are equal and ſimilar to them in the ſeveral ſolids [c]: and becauſe c 24. 11.
the ſolid angles at A, K, L are contained by three plane angles
equal to one another, each to each, the planes in which they are,
have the ſame inclination to one another [d]: For the ſame reaſon, d A. 11.



the other planes have the ſame inclination: Therefore the ſolids
LP, KR, AV are equal to one another [e]. What multiple ſoever, e B. 11.
therefore, the baſe LF is of the baſe AF, the ſame multiple is
the ſolid LV of the ſolid AV. Take, in like manner, EN the
leaſt multiple of EH, that is greater than EL, and complete the
parallelogram HS, and the ſolid HT; and it may be proved, as
before, that the ſolid ET is the ſame multiple of the ſolid ED,
that the baſe ES is of the baſe EC; conſequently the baſe HS
and the ſolid HT contain the baſe EC and the ſolid ED the ſame
number of times. And if the baſe LF be equal to the baſe HS,
the ſolid LV is equal [e] to the ſolid HT; and if LF be not equal
to HS, it is greater than HS, but leſs than ES; and the ſolid
LV is greater than the ſolid HT, but leſs than ET: Therefore
the baſes LF, HS contain the baſe EC the ſame number of times [f], f Def.A. 5.
and the ſolids LV, HT contain the ſolid ED the ſame number of
times: and the baſe HS and ſolid HT contain the baſe EC and
ſolid ED equally; therefore the baſe LF and ſolid LV contain
them equally: and the baſe LF and ſolid LV are any equimultiples
of the baſe AF and the ſolid AV; as many times, therefore,
as any multiple of the baſe AF contains the baſe EC, ſo
many times does the ſame multiple of the ſolid AV contain the

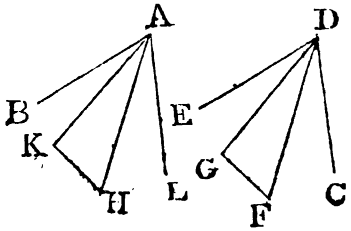
ſolid ED: Therefore [g], as the baſe AF is to the baſe EC, ſo is the ſolid AV to the ſolid ED. Wherefore, &c. Q. E. D.
g 5. Def. 5.

PROP. XXVI. PROB.

See N. AT a given point in a given ſtraight line, to make a ſolid angle contained by three plane angles equal, each to each, to the three plane angles which contain a given ſolid angle.

Let AB be a given ſtraight line, A a given point in it, and D a given ſolid angle contained by the three plane angles EDC, EDF, FDC; it is required to make, at the point A in the ſtraight line AB, a ſolid angle contained by three plane angles equal to the angles EDC, EDF, FDC.

a 11. 11. In the ſtraight line DF take any point F, from which draw [a]
FG perpendicular to the plane EDC, meeting that plane in G;
b 23. 1. join DG, and at the point A in the ſtraight line AB, make [b]
the angle BAL equal to the angle EDC, and in the plane BAL
make [b] the angle BAK equal to the angle EDG; then make AK
c 12. 11. equal to DG, and from the point K erect [c] KH at right angles
to the plane BAL; and make KH equal to GF, and join AH:
Then the ſolid angle at A is contained by the three plane angles BAL, BAH, HAL, which are equal, each to each, to the three plane angles EDC, EDF, FDC, which contain the ſolid angle at D.



Becauſe the angle BAL is
equal to EDC, and the angle
BAK to EDG, the angle KAL
is alſo equal to GDC: and
becauſe FG is perpendicular
to the plane EDC, the angle
d 3. Def. 11. FGD is a right angle [d]; and
the plane FGD is perpendi-
e 18. 11. cular to the plane EDC [e].
For the ſame reaſon, the angle HKA is a right angle, and the
plane HAK perpendicular to the plane BAL: and becauſe AK,
KH are equal to DG, GF, and they contain right angles; the
f 4. 1. baſe AH is equal [f] to FD, and the angle HAK to the angle to
GDF: and becauſe the two plane angles BAK, KAH are equal
to the two EDG, GDF; and that they are in planes which
have the ſame inclination to one another; therefore the third
Cor. A. 11. angle BAH is equal [g] to the third angle EDF. For the ſame
reaſon, the angle HAL is equal to the angle FDC. Wherefore,
at the point A in the ſtraight line AB a ſolid angle has been
made,

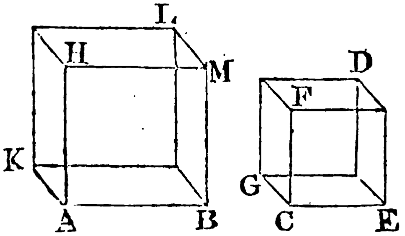
made, contained by the three plane angles BAL, BAH, HAL, which are equal to the three plane angles which contain the ſolid angle at D, each to each. Which was to be done.

PROP. XXVII. PROB.

To deſcribe from a given ſtraight line a ſolid parallelopiped, ſimilar and ſimilarly ſituated to one given.

Let AB be the given ſtraight line, and CD the given parallelopiped; it is required from AB to deſcribe a parallelopiped, ſimilar and ſimilarly ſituated to CD.

At the point A of the given ſtraight line AB, make [a] a ſolid angle contained by the plane angles BAK, KAH, HAB equal to the angles ECG, GCF, FCE, each to each; ſo that BAK be equal to the angle ECG, and KAH to GCF, and HAB to FCE: and as EC to CG, ſo make [b] BA to AK; and as GC to CF, ſo make [b] KA to AH; wherefore, by equality [c], as EC to CF, ſo is BA to AH: Complete the parallelogram BH, and the ſolid AL: and becauſe, as EC to CG, ſo BA to AK, the ſides about the equal angles ECG, BAK are proportionals; therefore the parallelogram BK is ſimilar to EG. For the ſame reaſon, the parallelogram KH is ſimilar to GF, and HB to FE: wherefore three parallelograms of the ſolid AL are ſimilar to three of the ſolid CD; and the three oppoſite ones in each ſolid are equal [d] and ſimilar to theſe, each to each, and the ſimilar planes have the ſame inclination to one another [e], becauſe the ſolid angles are contained by three plane angles equal to one another, each to each; therefore the ſolid AL is ſimilar [f] to the ſolid CD. Wherefore, from a given ſtraight line AB, a ſolid parallelopiped AL has been deſcribed ſimilar and ſimilarly ſituated to the given one CD. Which was to be done.



a 26. 11.

b 12. 6.

c 22. 5.

d 24. 11.

e A. 11.

f 9. Def. 11.

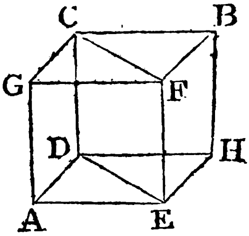
Book XI.

PROP. XXVIII. THEOR.

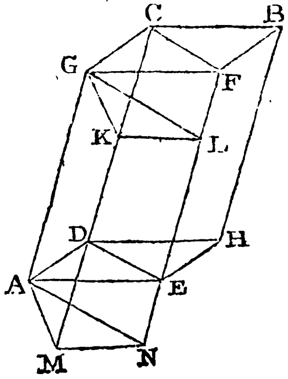
See N. IF a parallelopiped be cut by a plane paſſing through the diagonals of two of the oppoſite planes, it ſhall be cut in two equal parts.

Let AB be a parallelopiped, and DE, CF the diagonals which
join equal angles of the oppoſite parallelograms AH, GB; they
a 9. 11. are in the ſame plane, becauſe CD, EF are parallel [a], each of
them being parallel to AG: the plane CDEF ſhall cut the ſolid
AB into two equal parts.

Firſt, Let the ſtraight lines AG, DC, EF, HB be at right
angles to the planes BG, AH: and becauſe the triangle GCF is
b 34. 1. & equal and ſimilar to CBF [b], and the triangle
1. Def. 6. DAE to DHE; as alſo the rectangle
c 24. 11. GD to FH [c], and GE to CH:
and they have the ſame inclination to
d A. 11. one another [d], becauſe the ſolid angles
at G, B are contained by equal plane
angles; and they are ſituated in the ſame
order; therefore the priſm GCF-ADE
e B. 11. is equal [e] to the priſm BFC-HED.



But, let the ſtraight lines AG, DC,
EF, HB not be at right angles to the planes BG, AH: At the
f 23. 1. point G, in AG, make [f] in the plane GD the angle AGK equal
to HBF, and in the plane GE the angle AGL equal to HBC;
and make KM, LN each equal to AG, and join KL, AM, AN,
MN: and becauſe the angles AGK, AGL are equal to HBF,
HBC; and their planes have the ſame inclination [d], for the ſolid
angles at A, B are contained by equal plane angles; therefore
g Cor. A. the third angle KGL is equal [g] to the third angle CBF; and becauſe
11. the angle AGL is equal to
HBC. And the angle GLF is
h 29. 1. equal [h] to AGL, and GFL to
k 15. 11. HBC [k]; the angle GLF is equal
to GFL; and therefore the ſide
l 6. 1. GL is equal [l] to GF or CB. For
the ſame reaſon, GK is equal to
BF: and the angle KGL is equal
to CBF; therefore the triangle
KGL is equal and ſimilar to
m 4. 1. FBC [m], and AMN to HED; and
the parallelograms GM, GN, NK
are equal and ſimilar to BE, BD,
DF: and they are alike ſituated,



and have the ſame inclination to one another; therefore the priſm KGL-MAN is equal [c] to the priſm FBC-EHD: But the priſm CGF-DAE is equal [a] to the priſm KGL-MAN, becauſe their parallelograms have the ſide AG common, and their ſides oppoſite to it in the ſtraight lines CM, FN: Therefore alſo the priſm CGF-DAE is equal to the priſm FBC-EHD. Wherefore, &c. Q. E. D. c B. 11. a C. 11.

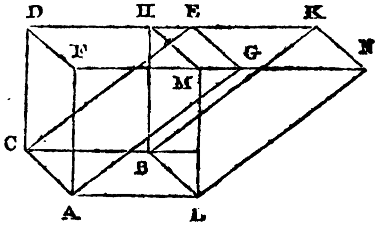
PROP. XXIX. THEOR.

SOLID parallelopipeds upon the ſame baſe, and of the ſame altitude, the ſides of which betwixt the baſe and oppoſite plane are terminated in the ſame ſtraight lines, are equal to one another. See N.

Let the parallelopipeds AH, AK be upon the ſame baſe AB, and of the ſame altitude, and let the ſtraight lines AF, AG, LM, LN be terminated in the ſame ſtraight line FN, and CD, CE, BH, BK be terminated in the ſame ſtraight line DK: the ſolid AH is equal to the ſolid AK.

Becauſe CH, CK are parallelograms, DC is equal and parallel to HB [a]; and CE to BK; therefore the angle DCE is equal to HBK [b]: and becauſe the two ſides DC, CE are equal to the two HB, BK, and the angle DCE to HBK; therefore the baſe DE is equal [c] to the baſe HK, and the triangle DCE equal and ſimilar to the triangle HBK; and the parallelogram DG to the parallelogram HN [d]. For the ſame reaſon, the triangle AFG is equal and ſimilar to LMN: and the parallelogram CF is equal and ſimilar to BM [e], and CG to BN, becauſe they are oppoſite: Therefore the priſm AFG-DCE is equal [f] to the priſm LMN-BHK.

a 34. 1.
b 29. 1.
c 4. 1.
d 36. 1.
e 24. 11.
f B. 11.

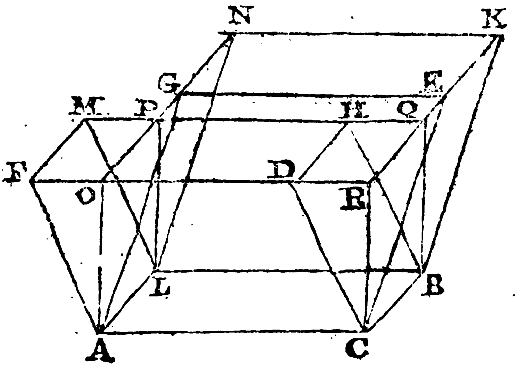


Take theſe equals from the whole ſolid ACBL-FDKN, and the remaining parallelopiped AK is therefore equal to the remaining parallelopiped AH. Wherefore, &c. Q. E. D.

PROP. XXX. THEOR.

SOLID parallelopipeds upon the ſame baſe, and of the ſame altitude, the inſiſting ſtraight lines * of which are not terminated in the ſame ſtraight lines, are equal to one another.

Let the parallelopipeds CM, CN be upon the ſame baſe AB, and of the ſame altitude; but their inſiſting ſtraight lines AF, AG, LM, LN, CD, CE, BH, BK not terminated in the ſame ſtraight lines; the ſolids CM, CN are equal.



Produce FD, MH, and NG, KE, and let them meet one another in the points O, P, Q, R; and join AO, LP, BQ, CR: and becauſe the plane LBHM is parallel to the oppoſite plane ACDF, and that the plane LBHM is that in which are the parallels LB, MPHQ, in which alſo is the figure BLPQ; and the plane ACDF is that in which are the parallels AC, FODR, in which alſo is the figure CAOR; therefore the figures BLPQ, CAOR are in parallel planes: In like manner, it may be ſhewn, that the figures ALPO, CBQR are in parallel planes: and the planes ACBL, ORQP are parallel; therefore the ſolid CP is a parallelopiped: But the ſolid CM, of which the baſe is ACBL, to which FDHM is the oppoſite parallelogram is equal [a] to the ſolid CP, of which the baſe is the parallelogram ACBL, to which

a 29. 11.

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* The inſiſting ſtraight lines are the ſides of the parallelograms betwixt the baſe and the oppoſite plane.

which ORQP is the one oppoſite, becauſe they are upon the ſame baſe, and their inſiſting ſtraight lines AF, AO, CD, CR; LM, LP, BH, BQ are in the ſame ſtraight lines FR, MQ: and the ſolid CP is equal [a] to the ſolid CN; for they are upon the ſame baſe ACBL, and their inſiſting ſtraight lines AO, AG, LP, LN; CR, CE, BQ, BK are in the ſame ſtraight lines ON, RK; therefore the ſolid CM is equal to the ſolid CN. Wherefore, ſolid parallelopipeds, &c. Q. E. D.

a 29. 11.

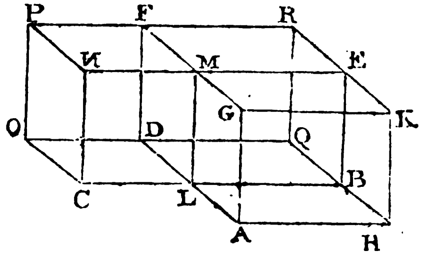
PROP. XXXI. THEOR.

SOLID parallelopipeds which are upon equal baſes, and of the ſame altitude, are equal.

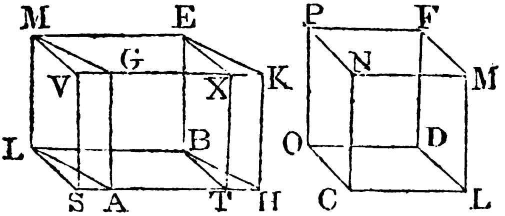
Let the ſolid parallelopipeds AE, CF be upon equal baſes AB, CD, and be of the ſame altitude; the ſolid AE is equal to the ſolid CF.

Firſt, Let the inſiſting ſtraight lines be at right angles to the baſes AB, CD, and let the baſes be equiangular, having the angle ALB equal to CLD, and let them be placed in the ſame plane, and ſo as that CL, LB be in a ſtraight line; therefore AL, LD are in a ſtraight line [a]; and the ſtraight line LM, which is at right angles to the plane in which the baſes are in the point L, is common [b] to the two ſolids AE, CF; let the other inſiſting lines of the ſolids be AG, HK, BE; DF, OP, CN. Produce OD, HB, and let them meet in Q, and complete the parallelopiped LR, of which the baſe is the parallelogram LQ, and LM is one of its inſiſting lines: Therefore, becauſe the parallelogram AB is equal to CD, as AB to LQ, ſo is CD to LQ [c]: and becauſe the parallelopiped AR is cut by the plane LMEB, which is parallel to the oppoſite planes AK, DR; as the baſe AB to the baſe LQ, ſo is [d] the ſolid AE to the ſolid LR: and becauſe the parallelopiped CR is cut by the plane LMFD parallel to the oppoſite planes CP, BR; as the baſe CD to the baſe LQ, ſo is [d] the ſolid CF to the ſolid LR: But as AB to LQ, ſo is CD to LQ, as before was proved; therefore as the ſolid AE to the ſolid LR, ſo is [e] the ſolid CF to the ſolid LR; and therefore the ſolid AE is equal [f] to the ſolid CF.

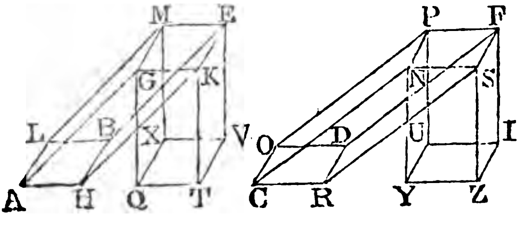
a 3. Cor. 15. 1.
b 13. 11.
c 7. 5.
d 25. 11.
e 11. 5.
f 9. 5.



Next, Let the inſiſting lines be at right angles to the baſes, but the baſes AB, CD not be equiangular. At the point L in
g 23. 1. the ſtraight line LB, make [g] the angle BLS equal to CLD, and let LS meet HA in S, and complete the parallelogram BS, and the parallelopiped SE, of which the baſe is SB, and one of the inſiſting lines is LM: Then, the ſolid SE is equal to the ſolid
h 29. 11. AE [h], becauſe they are upon the ſame baſe LE, and of the ſame altitude, and their inſiſting lines LS, LA, BT, BH; MV, MG, EX, EK are in the ſame ſtraight lines SH, VK: and be-
k 35. 1. cauſe the parallelogram AB is equal [k] to SB, for they are upon the ſame baſe LB, and between the ſame parallels LB, AT; and that the baſe AB is equal to the baſe CD; therefore SB is equal to CD; and the angle SLB is equal to the angle CLD: therefore, by the firſt caſe, the ſolid SE is equal to the ſolid CF: but the ſolid SE is equal to the ſolid AE, as was demonſtrated; therefore the ſolid AE is equal to the ſolid CF.



But, if the inſiſting lines AG, HK, BE, LM; CN, RS, DF, OP be not at right angles to the baſes AB, CD: From the



l 11. 11. points G, K, E, M; N, S, F, P draw [l] GQ, KT, EV, MX; NY, SZ, FI, PU perpendiculars to the plane in which are the baſes AB, CD, and let them meet it in the points Q, T, V, X; Y, Z, I, U, and join QT, TV, VX, XQ; YZ, ZI, IU, UY: Then, becauſe QG, KT are at right angles to the ſame plane,
m 6. 11. they are parallels [m]: and MG, EK are parallels; therefore the
n 15. 11. planes MQ, ET paſſing through them are parallel [n] to one another: For the ſame reaſon, the planes MV, GT are parallel;

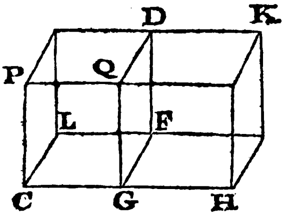
therefore the ſolid QE is a parallelopiped. In like manner, it may be proved, that the ſolid YF is a parallelopiped: But, from what has been demonſtrated, the ſolid EQ is equal to the ſolid FY, becauſe they are upon equal baſes MK, PS, and of the ſame altitude, and have their inſiſting lines at right angles to the baſes: and the ſolid EQ is equal [o] to the ſolid AE; and the o 29. or ſolid FY to the ſolid CF, becauſe they are upon the ſame baſes, 30. 11. and of the ſame altitude: Therefore the ſolid AE is equal to the ſolid CF. Wherefore ſolid parallelopipeds, &c. Q. E. D.

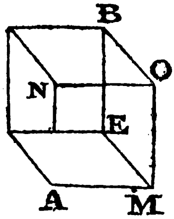
PROP. XXXII. THEOR.

SOLID parallelopipeds which have the ſame altitude, are to one another as their baſes.

Let AB, CD be ſolid parallelopipeds of the ſame altitude; they are to one another as their baſes; that is, as the baſe AE to the baſe CF, ſo is the ſolid AB to the ſolid CD.

Produce CG to H; and to FG apply the parallelogram FH equal [a] to AE, having FGH for one of its angles; and complete a Cor. 45. 1.





the parallelopiped GK upon the baſe FH, and having FD for one of its inſiſting lines; therefore the ſolids CD, GK are of the ſame altitude; therefore the ſolid AB is equal [b] to the ſolid GK, b 31. 11. becauſe they are upon equal baſes AE, FH, and are of the ſame altitude: and becauſe the ſolid CK is cut by the plane DG, which is parallel to its oppoſite planes, the baſe HF is to the baſe FC, as [c] the ſolid HD to the ſolid DC; But the baſe HF c 25. 11. is equal to the baſe AE, and the ſolid GK to the ſolid AB; therefore, as the baſe AE to the baſe CF, ſo is the ſolid AB to the ſolid CD. Wherefore, &c. Q. E. D.

Cor. 1. From this it is manifeſt, that priſms upon triangular baſes, of the ſame altitude, are to one another as their baſes.

Let the priſms AEM-NBO and CFG-PDQ have the ſame altitude; and complete the parallelograms AE, CF, and the parallelopipeds AB, CD, of which the inſiſting lines are MO, GQ:

and becauſe the parallelopipeds have the ſame altitude, AB is to CD, as the baſe AE to CF; wherefore the priſms, which are
d 28. 11. their halves [d], are to one another, as AE to CF; that is, as the triangle AEM to CFG.

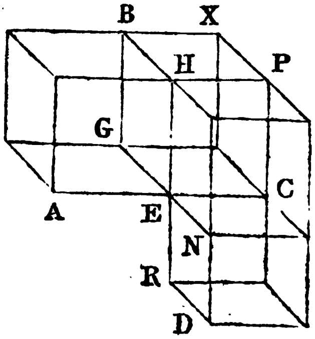
Cor. 2. Alſo, any priſms of the ſame altitude are to one another as their baſes: For priſms upon any other figures may be divided into priſms having triangular baſes: And each triangle is to each triangle, as the priſm upon the firſt to the priſm
e Cor. 24. 5. upon the other; therefore the whole priſm is to the whole as [e] the baſe of the firſt to the baſe of the other.

PROP. XXXIII. THEOR.

Similar ſolid parallelopipeds are one to another in the triplicate ratio of their homologous ſides.

Let AB, CD be ſimilar parallelopipeds, and the ſide AE homologous to the ſide CE: The ſolid AB has to the ſolid CD, the triplicate ratio of that which AE has to CE.

Let the baſes AG, CN be in the ſame plane, and the ſolids AB, CD on different ſides of it, and let AE, EC be in a ſtraight line; therefore, becauſe the angles AEG, CEN are equal, GE,
a 3. Cor. 15. 1. EN are in a ſtraight line [a]: And if the plane ED be produced beyond GN, it ſhall coincide with the plane GH, becauſe theſe planes have the ſame inclination to the common plane; therefore, becauſe the angle GEH is equal to REN, RE is in the ſame ſtraight line with EH [a]: complete the parallelogram GC and the parallelopipeds EX, NP upon the baſes GC, CN, ſo that EH be an inſiſting line in each of them, whereby they muſt be of the ſame altitude with the ſolid AB: and becauſe the ſolids AB, CD are ſimilar, as AE to EG, ſo is CE to EN; and, by alternation, as AE to EC, ſo is GE to EN: For the ſame reaſon, as AE to EC, ſo is HE to ER: But as AE to
b 1. 6. EC, ſo [b] is the parallelogram AG to GC, and the ſolid AB to
c 32. 11. the ſolid EX [c], becauſe they have the ſame altitude; and as GE to EN, ſo [b] is the baſe GC to the baſe CN, and the ſolid EX to the ſolid NP [c]: and as HE to ER, ſo [b] is the baſe HC to the baſe CR, and the ſolid NP to the ſolid CD [c]: But as AE to EC, ſo is GE to EN, and HE to ER; therefore, as the ſolid AB to
d 11. 5. the ſolid EX, ſo [d] is EX to NP, and NP to CD: But if four



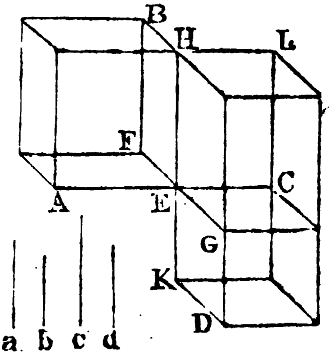
magnitudes be continual proportionals, the firſt is ſaid to have to the fourth the triplicate ratio of that which it has to the ſecond [e]; therefore the ſolid AB has to the ſolid CD, the triplicate ratio of that which AB has to EX: But as AB is to EX, ſo is AE to EC; therefore, the ſolid AB has to the ſolid CD, the triplicate ratio of that which the ſide AE has to the homologous ſide EC. Wherefore, ſimilar ſolids, &c. Q. E. D. e 11. Def. 5.

Cor. From this it is manifeſt, that, if four ſtraight lines be continual proportionals, as the firſt is to the fourth, ſo is the ſolid parallelopiped deſcribed from the firſt to the ſimilar ſolid ſimilarly deſcribed from the ſecond; becauſe the firſt ſtraight line has to the fourth the triplicate ratio of that which it has to the ſecond.

PROP. D. THEOR.

SOLID parallelopipeds contained by parallelograms equiangular to one another, each to each, have to one another the ratio which is compounded of the ratios of their ſides.

Let AB, CD be parallelopipeds, of which AB is contained by the parallelograms AF, FH, AH equiangular, each to each, to the parallelograms CG, GK, CK which contain the ſolid CD: The ratio which the ſolid AB has to the ſolid CD is the ſame with that which is compounded of the ratios of the ſides AE to EC, EF to EG, and EH to EK.



Let the ſolids AB, CD be placed on different ſides of the ſame plane, ſo that the baſes AF, CG be in the ſame plane, and AE, EC in the ſame ſtraight line; and becauſe the planes FH, GK have the ſame [a] inclination to the common plane, they coincide with one another; therefore, becauſe the angles AEF, AEH are equal to CEG, CEK, the ſtraight line EG is in the ſame ſtraight line with EF, and KE with EH: Complete the parallelogram GC and the ſolid parallelopiped GL, of which the baſe is GC and HE one of its inſiſting lines. Take any ſtraight line a, and as AE to EC, ſo [b] make a to b; and as FE to EG, ſo make b to c; and as HE to EK, ſo make c to d: Then, becauſe the parallelogram AF is equiangular to CG, AF is to CG, as a to c [c]; but as AF to CG,

a A. 11.
b 12. 6.
c 23. 6.

Book XI. ſo [d] is the ſolid AB, to the ſolid GL, becauſe they are of the
ſame altitude; therefore the ſolid AB is to the ſolid GL, as a to
d 32. 11. c: and the ſolid GL is to the ſolid CD, as [d] the baſe GH is to
the baſe GK; that is, as the ſtraight line EH to EK, or as c to
e 22. 5. d; therefore, by equality [e], the ſolid AB is to the ſolid CD, as
the ſtraight line a is to d. But the ratio of a to d is ſaid to be
f Def. A. 5. compounded [f] of the ratios of a to b, b to c, and c to d, which
are the ſame with the ratios of the ſides AE to EC, FE to EG,
and HE to EK: Therefore the ſolid AB has to the ſolid CD the
ratio which is the ſame with that which is compounded of the
ratios of the ſides AE to EC, FE to EG, and HE to EK.
Wherefore, &c. Q. E. D.

Cor. 1. Hence it is manifeſt, that equiangular parallelopipeds have to one another the ratio which is compounded of the ratios of their baſes, and of their inſiſting lines. For the ratio of AB to CD is compounded of the ratios of AB to GL, and GL to CD; that is, of the ratios of the baſe AF to the baſe GC, and of the inſiſting line HE to the inſiſting line EK.

Cor. 2. Hence alſo, any parallelopipeds have to one another the ratio which is compounded of the ratios of their baſes and of their altitudes. For if they be contained by rectangles, they are equiangular, and their inſiſting lines are the ſame with their altitudes: and if they be contained by any other parallelograms, they have the ſame ratio with thoſe contained by rectangles, which are upon equal baſes, and of the ſame altitudes with them.

PROP. XXXIV. THEOR.

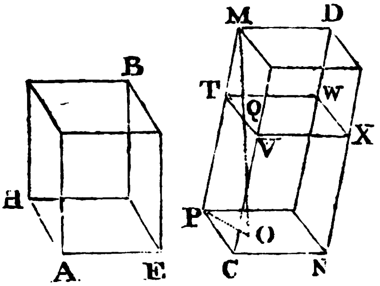
See N. THE baſes and altitudes of equal ſolid parallelopipeds, are reciprocally proportional; and if the baſes and altitudes be reciprocally proportional, the ſolid parallelopipeds are equal.

Let AB, CD be equal parallelopipeds; their baſes are reciprocally proportional to their altitudes; that is, as the baſe EH is to the baſe NP, ſo is the altitude of the ſolid CD to the altitude of the ſolid AB.

If the altitudes be equal, the ſolids are to one another as
a 32. 11. their baſes [a]: and the ſolids are equal; therefore the baſes are
b A. 5. equal [b].

But let the altitudes not be equal, and let the altitude of
CD be the greater: and if MP be at right angles to the baſe
PN, it is the altitude of CD: make PT equal to the altitude of
AB. But, if PM be not at right angles to the baſe PN, from
c 11. 11. the point M draw MO perpendicular to the plane PN [c], and let
it

it meet that plane in O; then MO is the altitude of the ſolid
CD: make OQ equal to the altitude of AB; and join OP, and
draw QT parallel to OP [d]: Therefore MP is to PT, as MO to d 31. 1.
OQ [e]; that is, as the altitude of CD to the altitude AB. Draw e 2. 6.
TV parallel to PC [d]: and let the plane QTV cut the planes of
the ſolid CD in the ſtraight lines TW, WX, VX: and becauſe
QT, TV are parallel to OP, PC, the plane TX is parallel to the
plane PN [f]; therefore CW is a parallelopiped [g]: and it is of f 15. 11.
the ſame altitude with the g Def. 4. 11.
ſolid AB; therefore, as the
ſolid AB to the ſolid CW,
ſo is the baſe EH to the
baſe PN [a]: and becauſe
the ſolid CD is cut by the
plane TX, which is parallel
to its oppoſite planes,
as the ſolid DT is to the
ſolid CW, ſo [h] is the baſe h 25. 11.
MV to the baſe VP, and
the ſtraight line MT to
the ſtraight line TP [k]: and, by compoſition [l], as the ſolid CD k 1. 6.
to the ſolid CW, ſo is MP to PT, that is, the altitude of CD l 18. 5.
to the altitude of AB: But the ſolid AB is to the ſolid CW, as
the ſolid CD to the ſolid CW, becauſe the ſolids AB, CD are
equal: Wherefore, as the baſe HE to the baſe PN, ſo [n] is the n 11. 5.
altitude of the ſolid CD to the altitude of the ſolid AB.



Next, Let the baſes of the ſolids AB, CD be reciprocally proportional to their altitudes; that is, as the baſe EH to the baſe NP, ſo is the altitude of the ſolid CD to the altitude of the ſolid AB: the ſolid AB is equal to the ſolid CD.

If the altitudes be equal, the baſes are alſo equal [b]; and there- b A. 5.
fore the ſolids are equal [m]. But, if the altitudes be not equal, m 31. 11.
let CD have the greater altitude: and the ſame conſtruction
being made, it may be demonſtrated, as before, that the ſolid
AB is to the ſolid CW, as the baſe HE to the baſe PN; and
that the ſolid CD is to the ſolid CW, as the altitude of the
ſolid CD to the altitude of the ſolid AB: But as the baſe HE
to the baſe PN, ſo is the altitude of the ſolid CD to the altitude
of the ſolid AB; therefore, as the ſolid AB to the ſolid CW,
ſo [n] is the ſolid CD to the ſolid CW. Wherefore the ſolid AB n 11. 5.
is equal [p] to the ſolid CD. Therefore, &c. Q. E. D. p 9. 5.

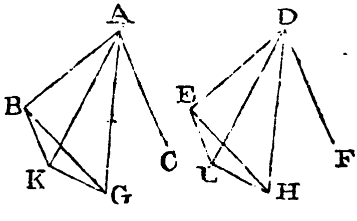
PROP. XXXV. THEOR.

See N. IF from the vertices of two equal plane angles, equal ſtraight lines be elevated above their planes, making equal angles with their ſides, each to each; the perpendiculars drawn from the extremities of the equal lines to the planes of the firſt angles, ſhall be equal to one another.

Let BAC, EDF be equal plane angles, and AG, DH equal ſtraight lines elevated above the planes BAC, EDF, making the angle BAG equal to EDH, and GAC to HDF; and from G, H let GK, HL be drawn perpendicular to the planes BAC, EDF, meeting them in K, L: the perpendicular GK is equal to HL.

a 12. 1. From the points K, L draw [a] KB, LE perpendiculars to AB, DE; and join BG, AK; EH, DL: and becauſe GK is perpendicular to the plane BAC; it makes right angles with AK, b 3. Def. 11. KB [b]; and the plane GKB paſſing through it is perpendicular c 18. 11. to the plane BAC [c]; therefore AB, which is at right angles to their common ſection BK, is at right angles to the plane BKG [d]; d 4. Def. 11. and ABG is therefore a right angle [d].

For the ſame reaſons, the angles HLD, HLE, HED are right angles: and becauſe the angle BAG is equal to EDH, and ABG, DEH are right angles; the two triangles ABG, DEH have two angles equal to two; and the ſides AG, DH oppoſite to equal angles, are alſo equal, therefore the other ſides AB, e 26. 1. BG are equal [e] to DE, EH: and becauſe the ſolid angles at A, D are contained by three plane angles equal to one another, each to each; the planes in f A. 5. which they are have the ſame inclination to one another [f]: g 6. Def. 11. and the angle GBK is the inclination [g] of the planes BAG, BAC, becauſe BG, BK are at right angles to their common ſection AB: and, for the ſame reaſon, the angle HEL is the inclination of the planes EDH, EDF; therefore the angle GBK is equal to HEL: and BKG, ELH are right angles; therefore, in the triangles GBK, HEL, there are two angles equal to two; and the ſides BG, EH oppoſite to equal angles, are equal; therefore the other ſides are equal [e], each to each: Wherefore the perpendicular GK is equal to the perpendicular HL. Therefore, &c. Q. E. D.



Cor. Likewiſe, if from the vertices of two equal plane angles, ſtraight lines be elevated, making equal angles with their ſides; they ſhall have the ſame inclination to the planes of the firſt angles.

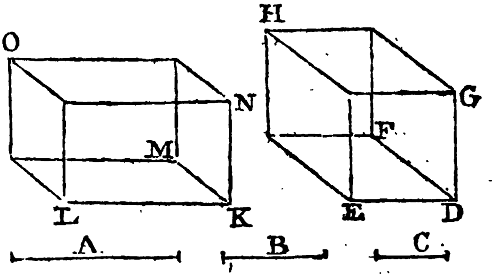
Let the ſtraight lines AG, DH be elevated above the planes of the equal angles BAC, EDF, and making the angle BAG equal to EDH, and the angle GAC to HDF. And take AG equal to DH, and draw GK, HL perpendiculars to the planes BAC, EDF, and join AK, DL: the angles GAK, HDL are the inclination [m] of the ſtraight lines AG, DH to the planes BAC, EDF: The angle GAK is equal to the angle HDL. m 5. Def. 11.

Draw [a] KB, LE perpendiculars to AB, DE; and join BG, EH: and it may be proved, as in the propoſition, that AB, BK are equal to DE, EL: and they contain right angles; therefore the baſe AK is equal [h] to DL: and AG is alſo equal to DH; therefore the two ſides GA, AK are equal to the two HD, DL: and the baſe GK is alſo equal to the baſe HL; therefore the angle GAK is equal [k] to the angle HDL. Q. E. D. a 12. 1. h 4. 1. k 8. 1.

PROP. XXXVI. THEOR.

If three ſtraight lines be proportionals, the ſolid parallelopiped deſcribed from all three as its ſides, is equal to the equilateral parallelopiped deſcribed from the mean proportional, equiangular to the other.

Let A, B, C be three proportionals; that is, A to B, as B to C: The ſolid deſcribed from A, B, C is equal to the equilateral ſolid deſcribed from B, contained by parallelograms equiangular to thoſe of the other figure.



Take a ſolid angle D contained by three plane angles EDF, FDG, GDE; and make each of the ſtraight lines ED, DF,

DG equal to B, and complete the parallelopiped DH: Make
LK equal to A, and at K, in LK, make [a] a ſolid angle contained
a 26. 11. by three plane angles LKM, MKN, NKL equal to the angles
EDF, FDG, GDE, each to each; and make KN equal to B,
and KM equal to C, and complete the parallelopiped KO: and
becauſe, as A is to B, ſo is B to C, and that A is equal to KL,
and B to each of the ſtraight lines DE, DF, and C to KM;
therefore LK is to DE, as DF to KM; that is, the ſides about
the equal angles are reciprocally proportional; therefore the
b 14. 6. parallelogram LM is equal [b] to EF: and becauſe EDF, LKM
are two equal plane angles, and the equal lines DG, KN are
drawn from their vertices above their planes, and containing
equal angles with their ſides; therefore the perpendiculars from
c 35. 11. G, N to the planes EDF, LKM are equal [c]; and the ſolids KO,
DH are therefore of the ſame altitude; and they are upon equal
d 31. 11. baſes LM, EF; therefore they are equal [d] to one another: But
the ſolid KO is deſcribed from the three ſtraight lines A, B, C,
and the ſolid DH from the ſtraight line B. If, therefore, three
ſtraight lines, &c. Q. E. D.

PROP. XXXVII. THEOR.

IF four ſtraight lines be proportionals, the ſimilar ſolid parallelopipeds ſimilarly deſcribed from them ſhall alſo be proportionals. And if the ſimilar parallelopipeds ſimilarly deſcribed from four ſtraight lines be proportionals, the ſtraight lines ſhall be proportionals.

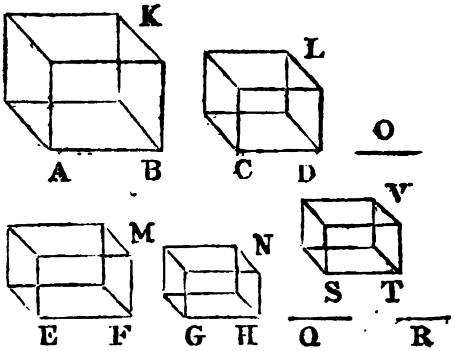
Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz. AB to CD, as EF to GH; and let the ſimilar parallelopipeds AK, CL, EM, GN be ſimilarly deſcribed from them; AK ſhall be to CL, as EM to GN.

a 11. 6. Make [a] AB, CD, O, P continual proportionals, as alſo EF,
GH, Q, R: and becauſe AB is to CD, as EF to GH; and that
b 11. 5. CD is to O, as [b] GH to Q, and O to P, as Q to R; therefore,
c 22. 5. by equality [c], AB is to P, as EF to R: But as AB to P, ſo [d] is
d Cor. 33. 11. the ſolid AK to the ſolid CL; and as EF to R, ſo [d] is the ſolid
EM to the ſolid GN; therefore [b], as the ſolid AK to the ſolid
CL, ſo is the ſolid EM to the ſolid GN.

But let the ſolid AK be to the ſolid CL, as the ſolid EM to the ſolid GN; the ſtraight line AB is to CD, as EF to GH.

e 12. 6. Take AB to CD, as [e] EF to ST, and from ST deſcribe [f] the
f 27. 11. parallelopiped SV ſimilar and ſimilarly ſituated to EM or GN:

and becauſe AB is to CD, as EF to ST, and that from AB, CD the ſimilar parallelopipeds AK, CL are ſimilarly deſcribed; and in like manner, the ſolids EM, SV from EF, ST; therefore AK is to CL, as EM to SV: But, by the hypotheſis, AK is to CL, as EM to GN; therefore GN is equal [g] to SV: But it is likewiſe ſimilar and ſimilarly ſituated to SV; therefore the planes which contain the ſolids are ſimilar and equal, and their homologous ſides GH, ST are equal to one another: and becauſe, as AB to CD, ſo is EF to ST, and that ST is equal to GH; AB is to CD, as EF to GH. Therefore, if four ſtraight lines, &c. Q. E. D. g 9. 5.



PROP. XXXVIII. THEOR.

IF a plane be perpendicular to another plane, a ſtraight line drawn from a point in one of them perpendicular to the other, ſhall fall on the common ſection of the planes. See N.

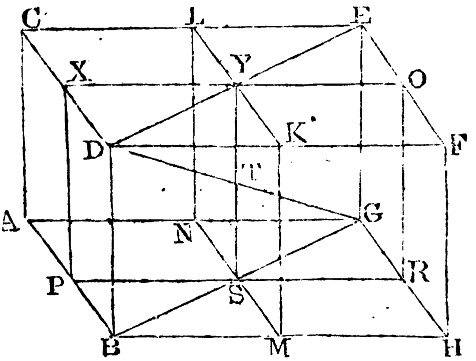
For a ſtraight line drawn from that point perpendicular to the common ſection of the planes is perpendicular to the other plane [a]; and only one ſtraight line can be drawn from that point perpendicular to that other plane [b]. a 4. Def. 11. b 13. 11.

PROP. XXXIX. THEOR.

IN a ſolid parallelopiped, if the ſides of two of the oppoſite planes be biſected, the common ſection of the planes paſſing through the points of diviſion, and the diameter of the parallelopiped biſect each other.

Let the ſides of the oppoſite planes CF, AH of the parallelopiped AF be biſected in the points K, L, M, N; X, O, P, R, and join KL, MN, XO, PR: and becauſe DK, CL are equal
a 33. 1. and parallel, KL is parallel [a] to DC: For the ſame reaſon, MN is parallel to BA: But BA is parallel to CD; therefore MN is
b 9. 11. parallel [b] to CD; and therefore alſo to KL. In like manner, it may be proved, that XO is parallel to PR. Let the planes paſſing through the parallels cut one another in the ſtraight lines YS; and let DG be the diameter of the parallelopiped AF: YS and DG do meet and biſect one another.

Join DE, BG: and becauſe DF, DC are doubles of DK, DX, the parallelograms CF, XK are ſimilar and ſimilarly ſituated, and having the angle at D common to both, they are about the ſame dia-
c 26. 6. meter [c]; therefore the diameter DE paſſes through the point Y: and becauſe DK is equal to KF,
d 2. 6. & DY is equal [d] to YE.
9. 5. For the ſame reaſon, BG paſſes through S, and BS is equal to SG: and becauſe DB is equal and parallel to CA, and CA to EG; therefore DB is equal and parallel [b] to EG: and DE, BG join their extremities; therefore DE is equal and parallel [a] to BG: and DG, YS are in the ſame plane with them; therefore they muſt meet one another; let them meet in T: and becauſe DE is parallel to BG, the alter-
e 29. 1. nate angles EDT, BGT are equal [e]; and the angle DTY is
f 15. 1. equal [f] to GTS: Therefore the triangles DTY, GTS have two angles of the one equal to two of the other, and the ſide DY equal to GS, which are oppoſite to equal angles; therefore the
g 26. 1. other ſides are equal [g], each to each. Wherefore DT is equal to TG, and YT equal to TS. Therefore, &c. Q. E. D.

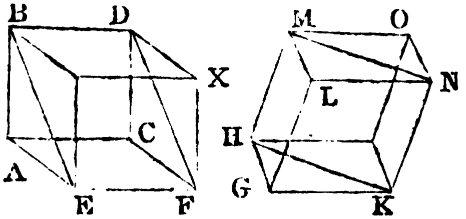


PROP.

PROP. XL. THEOR.

IF there be two triangular priſms of the ſame altitude, the baſe of one of which is a parallelogram, and the baſe of the other a triangle; if the parallelogram be double of the triangle, the priſms ſhall be equal to one another.

Let the priſm AEB-CFD ſtanding upon the parallelogram AF, and contained by the planes AEB, CFD, AD, DE, be of the ſame altitude with the priſm GHK-LMN ſtanding upon the triangle GHK, and contained by the planes GM, MK, KL, MLN: and let the parallelogram AF be double of the triangle GHK, the priſm AEB-CFD is equal to the priſm GHK-LMN.



Complete the ſolids AX, GO; and becauſe the parallelogram AF is double of the triangle GHK; and the parallelogram HK is double [a] of the ſame triangle; therefore the parallelogram AF is equal to HK. But parallelopipeds upon equal baſes, and of the ſame altitude, are equal [b] to one another; therefore the ſolid AX is equal to the ſolid GO: and the priſm AEB-CFD is the half [c] of the ſolid AX, and the priſm GHK-LMN the half [c] of the ſolid GO; therefore the priſm AEB-CFD is equal to the priſm GHK-LMN. Wherefore, &c. Q. E. D.

a 34. 1.
b 31. 11.
c 28. 11.

THE

THE

ELEMENTS

OF

GEOMETRY.

BOOK XII.

AXIOM.

Book XII.

AN arch of a circle is greater than its chord; but it is leſs than two ſtraight lines drawn from the ſame point, to touch the circle in the extremities of the arch.

PROP. I. THEOR *.

IF from the greater of two unequal magnitudes, there be taken more than its half, and from the remainder more than its half; and ſo on: There ſhall at length remain a magnitude leſs than the leaſt of the propoſed magnitudes.

Let AB and C be two unequal magnitudes, of which AB is the greater. If from AB there be taken more than its half, and from the remainder more than its half, and ſo on; there ſhall at length remain a magnitude leſs than C.

For C may be multiplied ſo, as at length to become greater than AB; let it be ſo multiplied, and let DE its multiple be

greater

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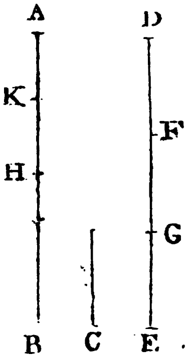
* This is Prop. I. Book X. of Euclid.

greater than AB, and let DE be divided into DF, FG, GE, each equal to C. From AB take BH greater than its half, and from the remainder AH take HK greater than its half, and ſo on, until there be as many diviſions in AB as there are in DE: and let the diviſions in AB be AK, KH, HB; and the diviſions in DE be DF, FG, GE: and becauſe DE is greater than AB, and that EG taken from DE is not greater than its half, but BH taken from AB is greater than its half; therefore the remainder GD is greater than the remainder HA: Again, becauſe GD is greater than HA, and that GF is not greater than the half of GD, but HK is greater than the half of HA; therefore the remainder FD is greater than the remainder AK: and FD is equal to C; therefore C is greater than AK; that is, AK is leſs than C. Q. E. D.



And if only the halves be taken away, the ſame thing may in the ſame way be demonſtrated.

PROP. II. THEOR.

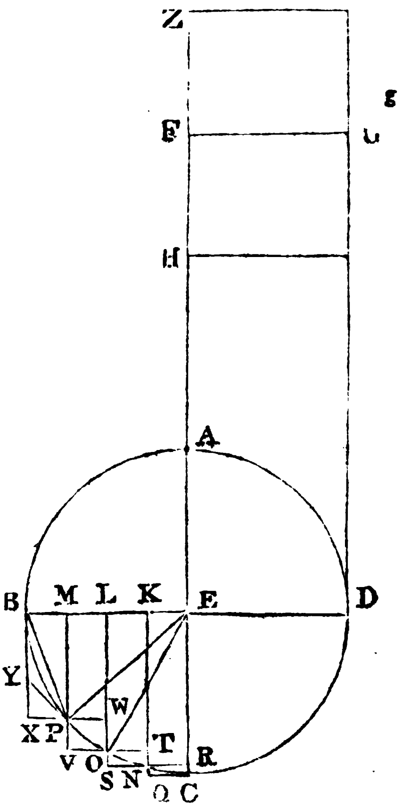
See N. EVERY circle is equal to the rectangle contained by its radius, and a ſtraight line equal to the half of its circumference.

Let ABCD be a circle, of which the diameter is AC, and the
centre E; and in AC produced, let EF be taken equal to the
a 11. 1. half of the circumference, and draw [a] the diameter BD at right
angles to AC, and complete the rectangle DF; the rectangle DF
is equal to the circle ABCD.

If not, it is either greater or leſs than the circle. Firſt, let
DF be greater than the circle; then ſome rectangle leſs than DF
is equal to the circle; let this be the rectangle DH: and divide
BE into ſuch equal parts EK, KL, LM, MB, that each of them
b {1. 12. 9. 6.} may be leſs than the fourth part of HF [b]; and draw [c] KN, LO,
MP parallel to AC, meeting the circumference in N, O, P; and
c 31. 1. complete the rectangles KC, EN, NL, KO, OM, LP, PB: and
d 36. 1. becauſe KE is equal to KL, the rectangle EN is equal [d] to NL:
For the ſame reaſon, KO is equal to OM, and LP to PB; therefore the rectangles NL, OM, PB are equal to the rectangles EN, KO, LP in the circle; and the rectangle KC is therefore the exceſs of the rectangles about the circle above thoſe within it: and if the ſame conſtruction be made in the other three ſectors, it may

may be proved in the ſame way, that all the circumſcribed rectangles exceed all the inſcribed ones by four times the rectangle KC: but becauſe FH is greater than four times EK, the rectangle HG is greater than four times KC; and HD being equal to the circle, is greater than the inſcribed rectangles; therefore DF is greater than the inſcribed rectangles, together with four times KC; that is, greater than the circumſcribed rectangles. Draw[e] PY touching the circle; and join PE; then the triangles EPY, EBY together are equal[f] to half the rectangle contained by EB and BY, YP together: But BY, YP are greater[g] than the arch BP; therefore the figure EBYP, and much more EBXP, is greater than the rectangle contained by EB and the half of the arch BP: In the ſame manner, it may be proved, that the quadrilateral EOVP is greater than the rectangle contained by EB and the half of the arch PO, and ſo on: Therefore all the circumſcribed rectangles together are greater than the rectangle contained by the radius and the half of the circumference; that is, greater than the rectangle DF: and it has been proved, that they are alſo leſs; which is impoſſible: Therefore the rectangle DF is not greater than the circle.

e 17. 3.
f 41. 1.
g Ax. 12.



Neither is it leſs: For, if poſſible, let the circle be equal to ſome rectangle DZ greater than DF: Then, if the ſame conſtruction be made, it may be proved, as before, that the exceſs of the circumſcribed rectangles above the inſcribed ones is leſs than GZ: and the circumſcribed rectangles are greater than the circle; that is, than DZ; therefore the inſcribed rectangles are greater than DF: Join BP; and the triangle EBP is leſs[f] than the rectangle contained by EB and the half of BP; and BP is leſs[g] than the arch BP; and the triangle EMP is leſs than EBP; therefore much

f 41. 1.
g Ax. 12.

Book XII. much more is the triangle EMP leſs than the rectangle contained by EB and the half of the arch BP. In the ſame manner, it may be proved, that the quadrilateral EPWO is leſs than the rectangle contained by EP and the half of the arch PO; and ſo on: Therefore all the inſcribed rectangles are leſs than the rectangle contained by the radius and the half of the circumference; that is, leſs than DF: and they were proved to be greater than it; which is impoſſible; therefore the rectangle DF is not leſs than the circle: and it was before proved, not to be greater than it; therefore DF is equal to the circle. Q. E. D.

Cor. 1. Hence it is manifeſt, that if a ſpace be leſs than a circle, rectangles can be inſcribed in the circle, which together are greater than that ſpace: and that if a ſpace be greater than a circle, rectangles can be deſcribed about the circle, which are together leſs than that ſpace.

Cor. 2. And conſequently, if a figure be greater than any ſeries of rectangles in a circle, but leſs than any ſeries of rectangles about it; that figure is equal to the circle.

PROP. III. THEOR *.

Circles are to one another as the ſquares of their diameters.

Let ABCD, EFGH be two circles, and BD, FH their diameters: as the ſquare of BD to the ſquare of FH, ſo is the circle ABCD to the circle EFGH.

Let the ſquare of BD be to the ſquare of FH, as the circle ABCD to ſome ſpace Z; this ſpace is equal to the circle EFGH.
a 11. 1. Draw [a] the diameters AC, EG at right angles to BD, FH; and
b 9. 6. divide [b] AK into any number of equal parts in M, N, O, and EL into as many in P, Q, R; and complete the rectangles KS, MT, NV; LW, PX, QY in the circles; and let the ſame conſtruction be made in the reſt of the ſectors: and becauſe AO,
c 15. 5. ER are the ſame parts of OC, RG; AO is to ER, as [c] OC to RG; therefore the ſimilar rectilineal figures upon them are pro-
d 22. 6. portionals [d]: and the ſquares of AO, ER are ſimilar figures [e] on
e 1. Def. 6. them; and the rectangles AO, OC and ER, RG are ſimilar figures [e] on OC, RG; therefore the ſquare of AO is to the ſquare of ER, as the rectangle AO, OC to the rectangle ER, RG: and AO is equal to ON, and ER to RQ; and the rect-
f 35. 3. angle AO, OC is equal [f] to the ſquare of OV, and the rectangle ER, RG to the ſquare of RY; therefore the ſquare of ON is to that

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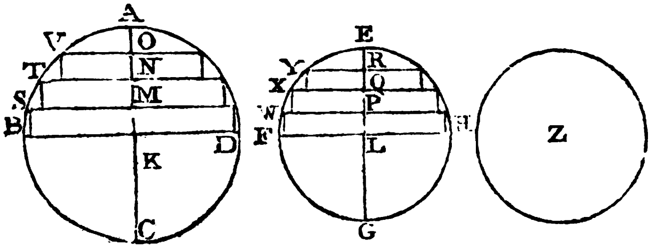
* This is Prop. II. Book XII. of Euclid.

that of RQ, as the ſquare of OV to that of RY; therefore NO
is to RQ as OV to RY; and, by alternation [g], NO is to OV,
as QR to RY; therefore the rectangle NV is ſimilar [e] to QY; g 16. 5.
therefore NV is to YQ, as [d] the ſquare of NO to the ſquare of e 1. Def. 6.
RQ; that is, as the ſquare of BD to the ſquare of FH [c], be- c 15. 5.
cauſe ON, RQ are the ſame parts of BD, FH. In the ſame
manner, it may be proved, that the other rectangles are to one
another as the ſquare of BD to that of FH: Wherefore all the



rectangles in the circle ABCD are to all the rectangles in the
circle EFGH as [h] the ſquare of BD to the ſquare of FH: But h 12. 5.
as the ſquare of BD to the ſquare of FH, ſo is the circle ABCD
to the ſpace Z; therefore, as the circle ABCD is to the ſpace Z,
ſo [k] are the rectangles in the circle ABCD to the rectangles in k 11. 5.
the circle EFGH: and the circle ABCD is greater than the rect-
angles in it; therefore the ſpace Z is greater [q] than the rectangles q 14. 5.
in the circle EFGH. In the ſame manner, it may be proved,
that the ſpace Z is leſs than any ſeries of rectangles about the
circle EFGH: Therefore the ſpace Z is equal [l] to the circle l 2. Cor.
EFGH: But the ſquare of BD is to the ſquare of FH, as the 2. 12.
circle ABCD to the ſpace Z; therefore the ſquare of BD is to
that of FH, as the circle ABCD to the circle EFGH. Q. E. D.

Cor. Hence, the circumferences of circles are to one another
as their diameters.

For a circle is equal [m] to the rectangle contained by the radius m 2. 12.
and half the circumference; therefore the rectangle contained by
BK and half the circumference ABCD is to the rectangle con-
tained by FL and half the circumference EFGH, as [n] the circle n 7. 5.
ABCD to the circle EFGH; that is, as [o] the ſquare of BD to o 3. 12.
that of FH, or as [c] the ſquare of BK to that of FL; and, alter-
nately [g], as the rectangle contained by BK and half the circum-
ference ABCD is to the ſquare of BK, ſo is the rectangle con-
tained by FL and half the circumference EFGH to the ſquare of
FL: and parallelograms of the ſame altitude are as their baſes [p]; p 1. 6.
therefore the half of the circumference ABCD is to BK, as half
the circumference EFGH to FL; and, alternately [g], as half the

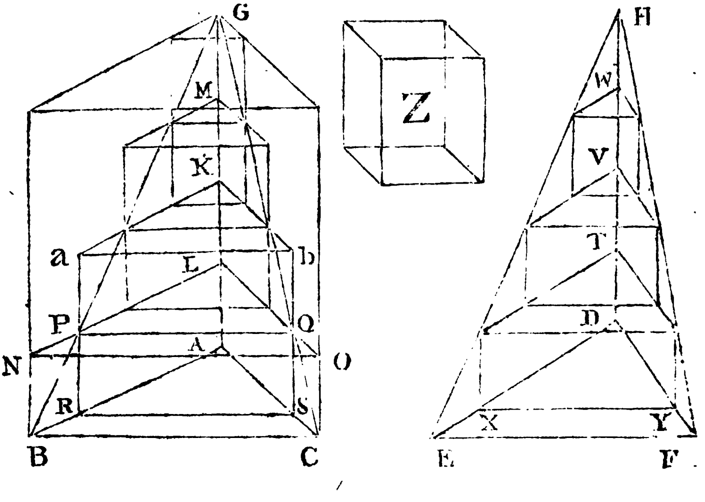
circumference ABCD to half the circumference EFGH, ſo is BK to FL; and taking their doubles, as the circumference ABCD
q 4. 5. to the circumference EFGH, ſo [q] is BD to FH.

PROP. IV. THEOR.

TRIANGULAR pyramids upon equal baſes, and of the ſame altitude, are equal to one another.

Let ABC-G, DEF-H be two pyramids of the ſame altitude, and having the equal triangles ABC, DEF for their baſes; the pyramid ABC-G is equal to the pyramid DEF-H.

If not, one of them is greater than the other: let the pyramid ABC-G be greater than the other pyramid by the ſolid Z: and complete the priſm, of which ABC is the baſe, and AG one



of its inſiſting lines: Then, if AG be biſected by a plane parallel
a Cor. 32. 11. to ABC, that plane will biſect [a] the priſm; for AG and the perpendicular from G to the plane ABC, are cut in the ſame ratio
b 17. 11. by that parallel plane [b]; therefore the parts are priſms on the ſame baſe, and having equal altitudes, and are therefore equal [a]. In the ſame manner, may the half of the remaining priſm be cut off; and ſo on: Therefore, if this be done continually, there
c 1. 12. ſhall at length remain a priſm leſs [c] than the ſolid Z: let this be the priſm ABC-LNO, and let LN, LO meet GB, GC in P,
d 31. 1. Q; and draw [d] PR, QS parallel to AG, and join PQ, RS;
e 13. Def. 11. therefore ARS-LPQ is a priſm [e] in the pyramid ABC-G: Let the

the other parts of AG equal to AL, be LK, KM, MG, and com-
plete priſms in the pyramid between the planes through L, K,
M: and let the ſame conſtruction be made in the pyramid
DEF-H: and becauſe LP is parallel [o] to AB, AG is to GL, as o 16. 11.
[f] AB to LP or AR: For the ſame reaſon, AG is to GL, as AC f 4. 6.
to AS; therefore BA is to AR, as [g] CA to AS; and the tri- g 11. 5.
angle ABC is ſimilar [h] to ARS: In the ſame manner, it may be h 6. 6.
proved, that the triangle DEF is ſimilar to DXY: and becauſe
AG, DH are equimultiples of AL, DT; as are alſo GL, HT;
therefore AG is to GL, as [k] DH to HT; wherefore BA is to AR, k 4. 5.
as ED to DX [g], and conſequently the triangle ABC is to ARS,
as [l] DEF to DXY: and the triangle ABC is equal to DEF; l 22. 6.
therefore the triangle ARS is equal [m] to DXY; and therefore m 14. 5.
the priſm LRS is equal [n] to the priſm TXY, for they are upon n Cor. 32.
equal baſes, and have the ſame part of the altitude of the pyra- 11.
mids for their altitude. In the ſame manner, may the other priſms in the two pyramids be proved to be equal, each to each; therefore all the priſms in the pyramid ABC-G together are equal to all the priſms in the pyramid DEF-H. Produce RP, SQ to a, b, and join ab; therefore LPQ-Kab is a priſm; and it is equal [n] to the priſm ASR-LPQ, becauſe they are upon the ſame baſe, and have equal altitudes: In the ſame manner, it may be proved, that the other inſcribed priſms are equal to the circumſcribed priſms ſtanding upon them: therefore all the inſcribed priſms, together with the priſm ABC-LNO, are equal to all the circumſcribed priſms; that is, they are greater than the pyramid ABC-G: But the priſms in the pyramid ABC-G are equal to thoſe in the pyramid DEF-H, and the ſolid Z is greater than the priſm ABC-LNO; therefore the priſms in the pyramid DEF-H, together with the ſolid Z, are greater than the pyramid ABC-G; that is, than the pyramid DEF-H, together with the ſolid Z: take away the common ſolid Z, and the priſms in the pyramid DEF-H are greater than the pyramid itſelf, and they are alſo leſs; which is impoſſible: Therefore the pyramid ABC-G is not unequal to the pyramid DEF-H, that is, it is equal to it. Q. E. D.

PROP. V. THEOR *.

EVERY triangular priſm may be divided into three equal triangular pyramids.

Let ABC-DEF be a triangular priſm, of which the baſe is the triangle ABC, and DEF the triangle oppoſite to it: It may be divided into three equal triangular pyramids.

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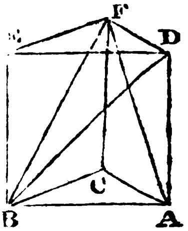
* This is Prop. VII. Book XII. of Euclid.

a 34. 1. b 4. 12. Join AF, FB, BD: and becauſe BD is the diameter of the parallelogram AE, the triangle ABD is equal [a] to EBD; and therefore the pyramid ABD-F is equal [b] to the pyramid EBD-F, for they are upon equal baſes, and of the ſame altitude: For the ſame reaſon, the pyramid ADF-B is equal to the pyramid ACF-B: But the pyramid ABD-F is the ſame with the pyramid ADF-B: Wherefore the pyramid ACF-B is equal to the pyramid EBD-F; therefore the priſm ABC-DEF is divided into three equal pyramids ABD-F, EBD-F, ACF-B. Wherefore, &c. Q. E. D.



Cor. 1. From this it is manifeſt, that every pyramid is the third part of a priſm which has the ſame baſe, and is of an equal altitude with it; for if the baſe of the priſm be any other figure than a triangle, it may be divided into priſms having triangular baſes.

c 2. Cor. 32. 11. Cor. 2.* Pyramids of equal altitudes are to one another as their baſes; becauſe the priſms upon the ſame baſes, and of the ſame altitude, are [c] to one another as their baſes.

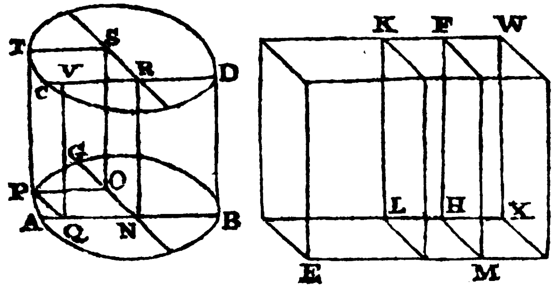
* *This is the 5th and 6th Propoſition of Book XII. of Euclid.*

PROP. VI. THEOR.

If a cylinder and a parallelopiped be upon equal baſes, and of the ſame altitude, they are equal to one another.

Let ABCD be a cylinder, and EF a parallelopiped of the ſame altitude; and let the circle ABG, which is the baſe of the cylinder, be equal to the baſe EH of the parallelopiped; the cylinder ABCD is equal to the parallelopiped EF.

a 33. 1. b 24. 11. If not, it is either greater or leſs than the parallelopiped: Firſt, let it be leſs, and let it be equal to the parallelopiped EK of the ſame altitude with EF, but having the baſe EL leſs than EH: and in the circle AGB let rectangles be made as in the ſecond propoſition, which together ſhall be greater than EL; and let the ſame be done in the oppoſite baſe CTD; and let NOPQ, RSTV be correſponding rectangles in them; therefore, becauſe the circles are equal, the rectangle PN is equal to RT, as was ſhown in Propoſition III; join NR, OS, PT, QV: and becauſe NQ is equal and parallel to RV, NR is equal and parallel to QV [a]: For the ſame reaſon, PT, OS are equal and parallel to NR or QV; therefore NT is a parallelopiped [b]: conſtruct, in the ſame manner, parallelopipeds upon the other rectangles in the circle AGB : and becauſe the parallelopipeds NT, EK are of the ſame altitude, NT is to EK, as [c] the baſe NP to the baſe EL : For the ſame reaſon, the parallelopipeds upon the other c 32. 11. rectangles in the circle AGB are each of them to EK as the rectangle upon which it ſtands to the baſe EL ; therefore all the parallelopipeds together are to EK as [d] all the rectangles in the d 2. Cor. circle AGB to the baſe EL : But the rectangles in the circle 24. 5. AGB are greater than EL ; therefore the parallelopipeds upon them are together greater than EK [e] : and EK is equal to the e A. 5. cylinder ABCD ; therefore the parallelopipeds upon the rectangles are greater than the cylinder ; and they are alſo leſs, becauſe they are contained in it ; which is impoſſible : Therefore the cylinder ABCD is not leſs than the parallelopiped EF.



Neither is it greater ; for, if poſſible, let the cylinder ABCD be equal to the parallelopiped EW of the ſame altitude with EF, but upon a baſe EX greater than EH ; that is, than the circle AGB : and becauſe EX is greater than the circle AGB, rectangles can be made about the circle, which together are leſs than EX [g] : and, if upon theſe rectangles be made parallelopipeds in g 1. Cor. the way that NT was made, it may be proved, as before, that 2. 12. they are together leſs than the parallelopiped EW ; that is, than the cylinder ABCD : and they are alſo greater, becauſe they contain the cylinder ; which is impoſſible ; therefore the cylinder is not greater than the parallelopiped EF : and it has been ſhown, that it is not leſs than it ; therefore the cylinder is equal to the parallelopiped. Wherefore, &c. Q. E. D.

Cor. 1. * Hence cylinders of the ſame altitude are to one another as their baſes, for the parallelopipeds upon baſes equal to them, and of the ſame altitude, are to one another as their baſes [c].

Cor. 2. Cylinders have to one another the ratio which is compounded of the ratios of their baſes, and of their altitudes. For this is the ratio of the parallelopipeds, which are upon equal baſes with them, and have the ſame altitudes [f]. f 2. Cor. D. 11.

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* *This is the 11th Propoſition of Book XII. of Euclid.*

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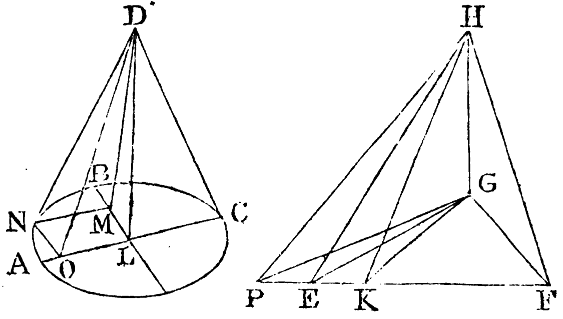
PROP. VII. THEOR.

IF a cone and a pyramid be upon equal baſes, and of the ſame altitude; they are equal to one another.

Let the cone of which the baſe is the circle ABC, and the vertex D, be of the ſame altitude with the pyramid of which the baſe is the figure EFG, and the vertex H; and let the circle ABC be equal to the figure EFG: the cone ABC-D is equal to the pyramid EFG-H.

If not, let them be unequal; and, firſt, let the cone be leſs than the pyramid; that is, let it be equal to ſome pyramid FGK-H of which the baſe FGK is leſs than FGE, or than the circle ABC; therefore, in the circle there can rectangles be made, which together are greater than FGK [a]: let LMNO be one of them, and join DL, DM, DN, DO: and becauſe the pyramids LMNO-D and FGK-H are of the ſame altitude, the pyramid LMNO-D is to the pyramid FGK-H, as [b] the baſe LMNO to the baſe FGK: In the ſame manner, if pyramids be erected on all the other rectangles in the circle ABC, it may be proved, that each of them is to the pyramid FGK-H, as the rectangle on which it ſtands to the baſe FGK: wherefore all the pyramids upon the rectangles are to the pyramid FGK-H, as [c] all the rectangles to the baſe FGK: But the rectangles together are greater than the baſe FGK; therefore the pyramids upon them are together greater than the pyramid FGK-H; that is, than the cone ABC-D: and they are alſo leſs, for they are contained in it; which is impoſſible; therefore the cone ABC-D is not leſs than



[a] 1. Cor. 2. 12.

[b] 2. Cor. 5. 12.

[c] 2. Cor. 24. 5.

than the pyramid EFG-H. Neither is it greater; for, if poſſible, let it be equal to ſome pyramid PFG-H, of which the baſe PFG is greater than EFG: and becauſe PFG is greater than the circle ABC, rectangles can be made about the circle which together are leſs [a] than PFG: and, if pyramids be erected upon theſe rectangles, having their common vertex at D, it may be proved, as before, that they are together leſs than the pyramid PFG-H; that is, than the cone ABC-D: and they are alſo greater, becauſe they contain the cone; which is impoſſible; therefore the cone ABC-D is not greater than the pyramid EFG-H: and it has been proved not to be leſs than it; therefore the cone ABC-D is equal to the pyramid EFG-H. Wherefore, &c. Q. E. D.

a 1. Cor. 2. 12.

Cor. 1. * Hence, a cone is the third part of a cylinder of the ſame altitude, and upon the ſame baſe with it; for the pyramid EFG-H is the third part of a priſm on the baſe EFG, and of the ſame altitude with it [d]: and this priſm is equal to a cylinder upon the baſe ABC and of the ſame altitude with the cone ABC-D [e]; therefore the cone is the third part of the cylinder.

d 1. Cor. 5. 12.
e 6. 12.

Cor. 2. Hence, cones of the ſame altitude, are to one another as their baſes.

* *This is the 10th Propoſition of Book XII. of Euclid.*

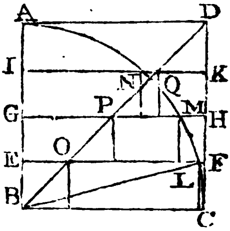
PROP. VIII. THEOR.

LET ABCD be a ſquare, and join BD, and from the centre B, at the diſtance BA, deſcribe the arch AC; and let any number of ſtraight lines EF, GH, IK be drawn parallel to BC, meeting the arch AC in L, M, N, and BD, in O, P, Q; and complete the rectangles BL, EM, GN in the circle, and the rectangles BO, EP, GQ about the triangle ABD: If, then, the figure thus conſtructed revolve about AB as an axis, the cylinders deſcribed by the rectangles in the ſector ABC, together with thoſe deſcribed by the rectangles about the triangle ABD, are equal to the cylinder deſcribed by the ſquare ABCD.

Join BL: and becauſe EO is parallel to AD, the angle EOB is equal [a] ADB, that is, to ABD [b]; therefore EB is equal [c] to EO: and becauſe BEL is a right angle, the ſquares of BE, EL are equal [d] to the ſquare of BL: and BL is equal to BC or EF; and BE to EO; therefore the ſquares of OE, EL are equal to the ſquare of EF; and becauſe the cylinders deſcribed by the revolution of BF and BL about the axis BE have the ſame altitude BE, they are to one another as [e] the circles deſcribed by the revolution of EF and EL about the point E: and theſe circles

a 29. 1.
b 5. 1.
c 6. 1.
d 47. 1.
e 1. Cor. 6. 12.

Book XII. circles are as the ſquares [f] of their diameters or radii; therefore the cylinder deſcribed by BL is to the cylinder deſcribed by BF, f 3. 12. as the ſquare of EL to that of EF: For the ſame reaſon, the cylinder deſcribed by BO is to the cylinder deſcribed by BF, as the ſquare of EO to that of EF; therefore the cylinders deſcribed by BL, BO together are to the cylinder deſcribed by BF, g 24. 1. as [g] the ſquares of EL, EO to the ſquare of EF: but the ſquares of EL, EO together are equal to the ſquare of EF; therefore the cylinders deſcribed by h A. 5. BL, BO are equal [h] to the cylinder deſcribed by BF. In the ſame manner, it may be demonſtrated, that the cylinders deſcribed by EM, EP are equal to that deſcribed by EH, and the cylinders deſcribed by GN, GQ equal to that deſcribed by GK: Wherefore all the cylinders deſcribed by the rectangles BL, EM, GN in the circle, and by BO, EP, GQ, ID about the triangle ABD, are together equal to the whole cylinder deſcribed by the ſquare ABCD. Q. E. D.



Cor. Hence, becauſe the hemiſphere deſcribed by the ſector ABC is greater than the cylinders in it, and the cone deſcribed by the triangle ABD leſs than the cylinders about it; it is manifeſt, that the hemiſphere, together with any ſeries of cylinders about the cone, is greater than the cylinder deſcribed by the ſquare AC: and that the cone, together with any ſeries of cylinders in the hemiſphere, is leſs than the ſaid cylinder deſcribed by AC.

PROP. IX. THEOR.

EVERY ſphere is two thirds of its circumſcribing cylinder.

Let ACR be the ſemicircle by the revolution of which about AR the ſphere is deſcribed; and from the centre B draw BC at right angles to AR, and complete the ſquare ABCD, and join BD: Then, if the whole revolve about AB, the ſector ABC ſhall deſcribe a hemiſphere, the ſquare AC a cylinder about the hemiſphere, and the triangle ABD a cone: The hemiſphere is two thirds of the cylinder deſcribed by AC; that is, becauſe a 1. Cor. 7. 12. the cone deſcribed by ABD is the third part [a] of the cylinder deſcribed by AC, the hemiſphere and the cone together are equal to the cylinder.

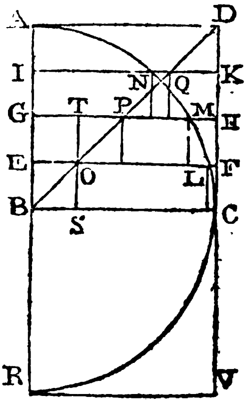
b 10. 1. If not, let them, firſt, be leſs than it by ſome ſolid X: and c 31. 1. biſect [b] AB in G, and draw [c] GH parallel to AD: Then, if AH, HB revolve about AB, the cylinders deſcribed by them ſhall be equal [d]; therefore the cylinder deſcribed by BH is the half of that deſcribed by AC: In the ſame manner, may the half of the cylinder deſcribed by AH be cut off from it; and ſo on: Therefore, if this be done continually, there ſhall at length remain a cylinder leſs [e] than the ſolid X*: let this be the cylinder deſcribed by AK; and let the other parts of AB equal to AI, be IG, GE, EB; and let the rectangles BO, EP, GQ be completed about the triangle ABD; and produce SO to T: and becauſe ES is equal to ET, the cylinders deſcribed by them are equal [d]: For the ſame reaſon, if the rectangles IP, AQ be completed, the cylinders deſcribed by them in the cone are equal to thoſe deſcribed about it by EP, GQ: conſequently, the cylinder deſcribed by AK is the exceſs of the cylinders about the cone above thoſe within it: But the cone is greater than the cylinders in it, and the ſolid X is greater than the cylinder deſcribed by AK; therefore the cone and X are greater than the cylinders about the cone: To theſe unequals, add the hemiſphere, and the hemiſphere and cone, with X, are greater than the hemiſphere and the cylinders about the cone: But the hemiſphere and cone, with X, are equal to the cylinder deſcribed by AC; therefore the cylinder deſcribed by AC is greater than the hemiſphere and the cylinders about the cone: and it is alſo leſs than them [f]; which is impoſſible: Therefore the cylinder deſcribed by AC is not greater than the hemiſphere and the cone.

Book XII.

d Cor. 6. 12.

e 1. 12.

f Cor. 8. 12.



Neither is it leſs; for, if poſſible, let it be leſs than them by ſome ſolid Z*: Then, as before, a rectangle may be cut off from AC, which ſhall deſcribe a cylinder leſs than the ſolid Z: let this be the rectangle EC: and, as before, conſtruct in the ſector ABC the rectangles BL, EM, GN: and it may be proved, as in the former caſe, that the cylinders deſcribed by theſe rectangles in the hemiſphere, together with Z, are greater than the cylinders about the hemiſphere, and therefore much more are they greater than the hemiſphere: To theſe unequals, add the cone; and the cylinders in the hemiſphere, together with the cone and Z, are greater than the hemiſphere and cone; that is, greater han the cylinder deſcribed by AC, together with Z: Take away the ſolid Z, and the cylinders in the hemiſphere, together with the cone, are greater than the cylinder deſcribed by AC: and they are alſo leſs [f]; which is impoſſible: Therefore the cylinder

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* The ſolids X and Z are not repreſented in the figure of this propoſition.

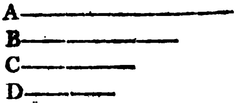
der deſcribed by AC is not leſs than the hemiſphere and the cone: and it has been ſhown, that it is not greater than them; therefore the cylinder is equal to the hemiſphere and the cone: a 1. Cor. 7. 12. But the cone deſcribed by ABD is the third part [a] of the cylinder; therefore the hemiſphere deſcribed by the ſector ABC is two thirds of the cylinder deſcribed by AC: Wherefore the whole ſphere is two thirds of the cylinder deſcribed about it by the rectangle AV. Therefore, &c. Q. E. D.

PROP. X. THEOR*.

SPHERES have to one another the triplicate ratio of that which their diameters have.

Let A be the diameter of the greater ſphere, and B the diameter of the leſſer; a 11. 6. and as A is to B, ſo make [a] B to C, and C b 11. Def. 5. to D; then the ratio of A to D is triplicate [b] of that of A to B: as A is to D, ſo is the greater ſphere to the leſſer.

Let cylinders be deſcribed about the ſpheres; then, the altitudes of the cylinders, and likewiſe the diameters of their baſes, are equal to A and B: and becauſe A is to B, as B to C, the c 1. Cor. 20. 6. ſquare of A is to that of B [c], as A to C: and circles are as the ſquares [d] of their d 3. 12. diameters; therefore, as A is to C, ſo is the baſe of the greater cylinder to that of the leſſer; and the altitude of the greater is to that of the leſſer as C to D: Wherefore the greater cy- e 2. Cor. 6. 12. linder has to the leſſer the ratio [e] which is compounded of the ratios of A to C, and of C to D; that is, the ratio [f] of A to D: f Def. A. 5. and each of the ſpheres is two thirds of its circumſcribing cy- h 9. 12. linder [h]; therefore, as A is to D, ſo [g] is the greater ſphere to the g C. 5. leſſer ſphere. Wherefore, &c. Q. E. D.



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* This is Prop. XVIII. Book XII. of Euclid.

THE

ELEMENTS

OF

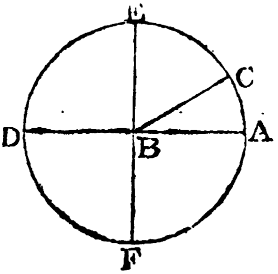
PLANE TRIGONOMETRY.

LEMMA I.

AN angle at the centre of a circle is to four right angles, as the arch on which it ſtands to the whole circumference.

Pl. Trig.

Let ABC be an angle at the centre B of the circle ACD, ſtanding on the arch AC. Draw BE at right angles to AB: and produce AB, EB to D, F: and becauſe the right angles at B are equal, the arches AE, ED, DF, FA are all equal [a]; therefore the circumference is four times the arch AE: and becauſe the angles ABC, ABE are at the centre, the angle ABC is to the right angle ABE, as [b] the arch AC to the arch AE: and quadrupling the conſequents, the angle ABC is to four right angles, as [c] the arch AC to four times the arch AE; that is, to the whole circumference.

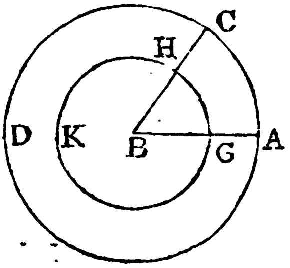


a 27. 3.
b 33. 6.
c Cor. 4. 5.

LEMMA II.

IN unequal circles, the arches, upon which equal angles at the centre ſtand, have the ſame ratio to the circumferences of the circles.

Let ABC, GBH be equal angles at the centre of the circles ACD, GHK, ſtanding upon the arches AC, GH: and becauſe ABC is an angle at the centre, it is to four right angles, as [a] the arch AC to the circumference ACD. For the ſame reaſon, the angle GBH or ABC is to four right angles, as the arch GH to the circumference GHK: Therefore the arch AC is to the circumference ACD, as [b] the arch GH to the circumference GHK.



a Lem. 1.
b 11. 5.

Note. The arches AC, GH are ſaid to be ſimilar, when they have the ſame ratio to the circumferences ACD, GHK.

LEMMA III.

IF two magnitudes conſiſt of parts each equal to the ſame magnitude; they have to one another the ſame ratio which the numbers denoting the parts of which they conſiſt, have to one another.

Let A, B conſiſt of parts each equal to C: and let I be unity, and M the number of parts equal to C, in A; and N the number of parts equal to C, in B: A is to B, as M to N.

Becauſe there are as many parts equal to C, in A, as there are units I in the number M; A, M are equimultiples of C, I; therefore A is to C, as [a] M to I: and B, N are equimultiples of C, I; therefore A is to B [b], as M to N.

a C. 5.
b Cor. 4. 5.

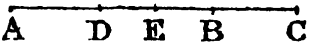
LEMMA IV.

HALF the ſum of two unequal magnitudes, together with half their difference, is equal to the greater; and half the difference taken from half the ſum is equal to the leſs.

Let

Pl. Trig.

Let AB be the greater of two unequal magnitudes, and BC the leſs; therefore AC is their ſum: make AD equal to BC; therefore BD is their difference: and if BD be biſected in E, BE or ED is half their difference: and becauſe AD is equal to BC, and DE to EB; the whole AE is equal to EC; therefore AE or EC is the half of the ſum AC. But the greater magnitude AB is equal to AE, EB together; that is, to half the ſum added to half the difference; and the leſs BC is equal to the exceſs of EC half the ſum, above EB half the difference.



DEFINITIONS.

I.

An arch of a circle is called the meaſure of the angle at the centre ſtanding on that arch. Thus, the arch AC is the meaſure of the angle ABC.

II.

The circumference of every circle is ſuppoſed to conſiſt of 360 equal parts, each of which is called a degree: and a degree is ſuppoſed to conſiſt of 60 equal parts, called minutes; and a minute of 60 equal parts, called ſeconds; and ſo on. Alſo, an angle at the centre is ſaid to be of as many degrees, minutes, &c. as there are in the arch, which is the meaſure of that angle.

Cor. 1. A right angle is an angle of 90 degrees. For the meaſure of it is the fourth part of the circumference.

Cor. 2. Arches of different circles, which meaſure the ſame angle, contain the ſame number of degrees, and parts of a degree. For the number of degrees and parts of a degree in each of them is to 360, as each of the arches to its circumference[a]; that is, as the angle which they meaſure to four right angles[b].

a Lem. 3.
b Lem. 1.

III.

The ſine of an arch is a ſtraight line drawn from one extremity of the arch perpendicular to the diameter paſſing through the other extremity of it. Thus, CD drawn from C perpendicular to AF, is the ſine of the arch AC.

Cor. 1. The ſine of AH, the fourth part of the circumference, is BH equal to the radius.

Cor. 2. The ſine of an arch is the half of the chord of twice that arch. For CM is biſected in D[c].

c 3. 3.

IV.

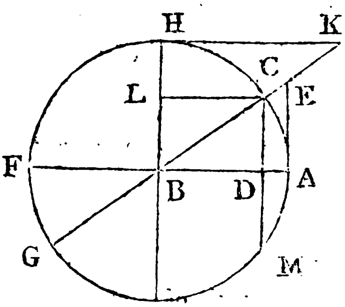
The verſed ſine of an arch is the ſegment of the diameter, to which the ſine is perpendicular, between the ſine and the arch. Thus, AD is the verſed ſine of the arch AC.

V.

Pl. Trig.

V.

The tangent of an arch is a ſtraight line touching the circle at one extremity, and meeting the diameter that paſſes through the other extremity. Thus, AE is the tangent of the arch AC.



Cor. The tangent of the eighth part of the circumference is equal to the radius.

VI.

The ſecant of an arch is the ſtraight line drawn from the centre to the fartheſt extremity of the tangent of that arch. Thus, BE is the ſecant of the arch AC.

VII.

The ſine, verſed ſine, tangent, and ſecant of an arch, are alſo called the ſine, verſed ſine, tangent, and ſecant of the angle, of which that arch is the meaſure. Thus, CD is the ſine of the angle ABC; AD is its verſed ſine; AE is its tangent; and BE its ſecant.

Cor. The radius is equal to the ſine, or verſed ſine of a right angle; or to the tangent of half a right angle.

VIII.

The difference between an arch and the half of the circumference, or between an angle and two right angles, is called the ſupplement of that arch, or of that angle. Thus, the arch CF is the ſupplement of the arch AC; and the angle CBF is the ſupplement of the angle ABC.

Cor. The ſine, tangent, and ſecant of any angle ABC, are alſo the ſine, tangent, and ſecant of its ſupplement: For, by Def. 4th, CD is the ſine of the arch CF, that is, of the angle CBF, by Def. 7th: and if CB be produced to G, it is manifeſt, from Def. 5th and 6th, that AE is the tangent, and BE the ſecant of the arch AG, that is, of the angle ABG or CBF, by Def. 7th.

IX.

The difference between an arch and the fourth part of the circumference, or between an angle and a right angle, is called the complement of that arch, or of that angle. Thus, if BH be at right angles to AB, the arch CH is the complement of the arch CA, and the angle CBH is the complement of the angle ABC, or of CBF.

X.

The ſine, tangent, or ſecant of the complement of any angle, is called the coſine, cotangent, or coſecant of that angle. Thus, if CL, HK be perpendiculars to BH; CL is the coſine of the angle ABC, and HK its cotangent, and BK its coſecant.

Pl. Trig.

Cor. 1. The ſegment of the diameter to which the ſine is perpendicular, between the ſine and the centre, is equal to the coſine. Thus, BD is equal to CL[d], the coſine of the angle ABC.

d 34. 1.

Cor. 2. The coſine of any arch is to the ſine, as the radius to the tangent of the ſame arch. For the triangles BDC, BAE are equiangular[e], becauſe the angle ABE is common, and BDC, BAE are right angles; therefore BD is to DC, as BA to AE[f].

e 32. 1.
f 4. 6.

Cor. 3. The radius is a mean proportional between the coſine and the ſecant of any angle. For BD is to BC or BA, as BA to BE.

Cor. 4. The radius is a mean proportional between the tangent and cotangent of any angle. For, the alternate angles ABE, BKH are equal[g]; and BAE, BHK are right angles; therefore the triangles BAE, BHK are equiangular; wherefore EA is to AB, as BH or BA to HK.

g 29. 1.

Cor. 5. The tangent and cotangent of any angle are reciprocally proportional to the tangent and cotangent of any other angle. For the tangent of the firſt angle is to the radius, as the radius to its cotangent; and the radius is to the tangent of the other angle, as the cotangent of that other angle to the radius: Therefore, by perturbate equality[h], the tangent of the firſt angle is to the tangent of the other, as the cotangent of the other to the cotangent of the firſt.

h 22. 5.

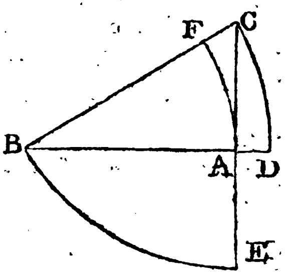
PROP. I.

IN a right angled triangle, if the hypothenuſe, or ſide oppoſite to the right angle, be made the radius of a circle; the other ſides are the ſines of the angles oppoſite to them, or the coſines of the angles adjacent to them. And if either of the ſides about the right angle be made the radius, the other ſide is the tangent of the angle oppoſite to it, and the hypothenuſe is the ſecant of the ſame angle.

Let ABC be a triangle, having the right angle BAC: and from the centre B at the diſtance BC, deſcribe a circle meeting BA produced in D: and becauſe AC is drawn from C the extremity of the arch CD, perpendicular to the radius BD, which paſſes through the other extremity D; CA is the ſine of the

Pl. Trig. the arch CD[a], or of the angle CBD[b] oppoſite to CA; and
BA the ſegment of the diameter, to which CA is perpendicular,
a 3. Def. between the ſine CA and the centre
b 7. Def. B, is the coſine of the angle ABC[c].
c 1. Cor. 10. Def. In the ſame manner, if from the centre
C, at the diſtance CB, the arch BE be
deſcribed, it may be proved, that BA
is the ſine, and AC the coſine of the
angle ACB.

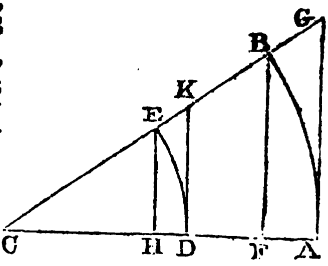
Again, from the centre B, at the
diſtance BA, deſcribe the arch AF.
And becauſe BAC is a right angle,
f Cor. 16. 3. AC touches the circle at A[f]; and it
meets the diameter paſſing through F, in C; therefore AC is
d 5. Def. the tangent[d] of the arch AF, or of the angle ABC: and BC,
which is drawn from the centre to the extremity C of the tan-
e 6. Def. gent, is the ſecant[e] of the ſame angle ABC.



PROP. II.

If two unequal arches meaſure the ſame angle, the ſine, verſed ſine, tangent, and ſecant of one of them, have to the radius of that arch, the ſame ratios, which the ſine, verſed ſine, tangent, and ſecant of the other arch, have to the radius of it.

Let AB, DE be two unequal arches, which meaſure the angle
a 11. or 12. 1. ACB: and draw[a] BF, AG, EH, DK at right angles to AC:
Then, BF is the ſine[b], AF the verſed ſine[c], AG the tangent[d],
b 3. Def. and CG the ſecant[e] of the arch AB: and EH is the ſine[b], HD
c 4. Def. d 5. Def. the verſed ſine[c], DK the tangent[d], and CK the ſecant[e] of the
e 6. Def. arch DE. And becauſe CFB,
CHE are right angles, and FCB
common to the triangles CFB,
f 32. 1. CHE; they are equiangular[f];
g 4. 6. and therefore FB is to BC, as[g]
HE to EC; that is, the ſine of
the arch AB is to the radius BC,
as the ſine of the arch ED to the
radius CE. Alſo CB or CA is
to CF, as CE or CD to CH;
h E. 5. and, by converſion[h], CA is to
AF, as CD to DH; that is, the radius CA is to the verſed ſine
AF, as the radius CD to the verſed ſine DH. In the ſame
manner, it may be proved, that the tangent AG is to the radius
AC,



AC, as the tangent KD to the radius DC; and that the ſecant CG is to the radius CA, as the ſecant CK to the radius CD. Pl. Trig.

Cor. 1. Hence, if the radius AC be divided into any number of equal parts, and the ſine BF be a multiple of one of them; the number of parts in BF is to the number in AC, as [k] BF to AC; that is, as EH to CD: Wherefore the numbers denoting the parts in BF and AC, expreſs the ratio, which the ſine of the angle ACB, in any circle, has to the radius of that circle. k 3. Lem.

In the ſame manner, the ratios which the ſines, verſed ſines, tangents, and ſecants of any angles, have to the radius, may be expreſſed in numbers, if they be multiples of any parts of the radius.

In the Trigonometrical Tables, the radius is ſuppoſed to be divided into 10,000,000, &c. equal parts: and the ſines, tangents, ſecants, and verſed ſines, are repreſented by the numbers, which ſhew what multiples of one of theſe parts are moſt nearly equal to them.

Cor. 2. Hence, the ſines of two arches in any circle are to one another, as the numerical expreſſions of the ſines of the angles meaſured by them, in the tables. For the ſine of the firſt arch is to the radius of the circle, as the numerical expreſſion of the ſine of the angle meaſured by that arch to the radius of the tables; and the radius of the circle is to the ſine of the other arch, as the radius of the tables to the numerical expreſſion of that other ſine: Therefore, by equality [l], the firſt ſine is to the other ſine, as the numerical expreſſion of the firſt to the numerical expreſſion of the other ſine. l 22. 5.

In the ſame manner, it may be proved, that the tangents, ſecants, and verſed ſines, of arches of the ſame circle, are to one another, as their numerical expreſſions.

Cor. 3. From this and the preceding propoſitions, it is manifeſt, that in a right angled triangle, the hypothenuſe is to a ſide, as the radius in the tables to the numercial expreſſion of the ſine of the angle oppoſite to that ſide: and that one of the ſides about the right angle is to the other, as the radius to the tangent of the angle oppoſite to that other; as alſo, that a ſide is to the hypothenuſe, as the radius to the ſecant of the angle contained by them.

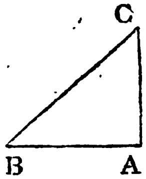
SOLUTION of the Caſes of Right-angled Triangles.

GENERAL PROPOSITION.

IN a right angled triangle, of the three ſides, and either of the acute angles, any two being given, the other two and the remaining acute angle may be found.

But when the two acute angles are given, the ſides cannot be found from them; but only their ratios, which are the ſame with thoſe of the ſines of their oppoſite angles.

When one of the acute angles is given, the other is alſo given, for it is the complement of the former; and therefore the ſine of one of them is the coſine of the other.



The ſeveral caſes of this propoſition may be reſolved by the help of the firſt and ſecond propoſitions, and their corollaries, as in the following table. Or, if two ſides be given, the third may be found by the 47th of the 1ſt of the Elem. For the ſquare of BC is equal to the ſquares of BA, AC; and therefore the ſquare of AB is equal to the exceſs of the ſquare of BC above the ſquare of AC.

In the Tables, R. ſignifies the radius, ſin. a ſine, tan. a tangent, ſec. a ſecant, cos. a coſine, cot. a cotangent; and R^2. ſignifies the ſquare of the radius, ſin ^{2}. the ſquare of the ſine, $\frac{1}{2}$A the half of A. Alſo R : ſin. B : : BC : CA: as radius to the ſine of the angle at B, ſo is the ſide BC to the ſide CA; and 38° 42′ 12″ 38 degrees, 42 minutes, 12 ſeconds.

Caſes.	Given.	Sought.	Solution.
1	BC and B, the hypot. and an angle.	AC BA	R : ſin. B : : BC : CA. R : cos. B : : CB : BA.
2	AB and B, a ſide and an angle.	AC	R : tan. B : : BA : AC.
3		BC	R : ſec. B : : AB : BC.
4	CB and BA, the hypot. and a ſide.	C AC	CB : BA : : R : ſin. C. R : cos. C : : BC : CA.
5	BA and AC, the two ſides.	B BC	BA : AC : : R : tan. B. R : ſec. B : : AB : BC.

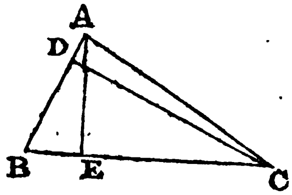
PROP.

PROP. III.

IN any triangle, the ſides are to one another, as the ſines of the angles oppoſite to them.

If one of the angles ABC be a right angle; by making AC the radius, the ſides AB, BC are the ſines [a] of the angles at C, A oppoſite to them; and the radius AC is the ſine [b] of the right angle ABC.

But, if none of them be a right angle; draw [c] CD, AE perpendiculars to AB, BC: therefore, in the right angled triangles ADC, AEC, if the hypothenuſe AC be made the radius, the ſide CD is the ſine [a] of the angle BAC, and AE is the ſine of the angle ACB: and becauſe the angles at D, E are right angles and ABC is common to the triangles ABE, CBD; they are equiangular [d]; and therefore CB is to CD, as [e] BA to AE; and, alternately, CB is to BA, as [f] CD to AE; that is, as the ſine of the angle BAC to the ſine of the angle ACB.



a 1. P. T.
b Cor. 7. Def.
c 12. 1.
d 32. 1.
e 4. 6.
f 16. 5.

PROP. IV.

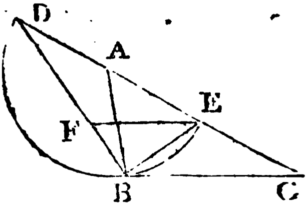
IN any triangle, the ſum of two ſides is to their difference, as the tangent of half the ſum of the angles at the baſe, to the tangent of half their difference.

Let ABC be a triangle, of which the ſides AC is greater than AB; the ſum of CA, AB is to their difference, as the tangent of half the ſum of the angles ABC, ACB to the tangent of half their difference.

From the centre A, at the diſtance AB, deſcribe the ſemicircle DBE, meeting AC in E, and CA produced in D; therefore DC is the ſum of CA, AB, and CE is their difference: join BD, BE; and through E draw [a] EF parallel to CB; therefore the angle FEB is equal [b] to EBC: and becauſe the angle BAD is equal [c] to the angles ABC, ACB, and that the angle BED at the circumference is half [d] of the angle BAD; the angle BED is half the ſum of the angles ABC, ACB: and becauſe AEB is equal [c] to the angles EBC, ECB, and ABE is equal [e] to AEB, for AE is equal to AB; the angle ABE is equal to the angles

a 31. 1.
b 29. 1.
c 32. 1.
d 20. 3.
e 5. 1.

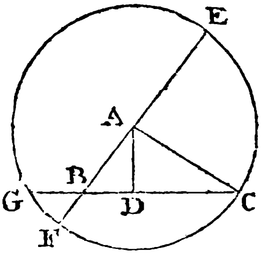
ECB, EBC: to each of theſe equals add the angle EBC; and the whole angle ABC is equal to ACB and twice EBC; therefore twice EBC is the difference between the angles ABC, ACB; and EBC or BEF is half their difference: but becauſe the angle EBD in a ſemicircle is a right f 31. 3. angle [f]; if BE be made the radius, g 5. Def. BD is the tangent [g] of BED, and BF the tangent of BEF: and becauſe FE is parallel to BC; CD h 2. 6. is to CE, as BD to BF [h]: Wherefore, as the ſum of the ſides CA, AB to their difference, ſo is the tangent of half the ſum of the angles ABC, ACB to the tangent of half their difference.



PROP. V.

IN any triangle, if a perpendicular be drawn to the baſe from the oppoſite angle; the baſe is to the ſum of the ſides, as the difference of the ſides to the exceſs of twice the greater ſegment above the baſe.

Let ABC be a triangle, of which the ſide AC is greater than AB; and from A, draw [a] AD perpendicular to BC; and from the centre A, at the diſtance AC, deſcribe the circle CEF, meeting AB produced in E, F, and CB produced in G: and becauſe AD, which paſſes through the centre, cuts CG at right angles in D; GD is equal to DC; therefore DC adjacent to the greater ſide AC is the greater ſegment: as the baſe CB to the ſum of the ſides CA, AB, ſo is the difference of CA, AB to the exceſs of twice CD above the baſe CB.



Becauſe EA, AF are each of them equal to AC; BE is the ſum of CA, AB, and BF their difference: and becauſe CG is double of CD, GB is the exceſs of twice CD above the baſe CB: But becauſe CG, EF in the circle, cut one another in B, the b 35. 3. rectangle EB, BF is equal [b] to the rectangle CB, BG; wherefore c 16. 6. CB is to BE, as [c] BF to BG. Therefore, &c. *

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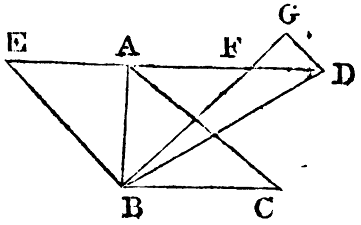
* N.B. The foregoing Propoſitions are ſufficient for ſolving the caſes; but the following are often uſed.

PROP. VI.

Pl. Trig.

IN any triangle, if any angle be found ſuch, that the radius is to its tangent, as one of the ſides to the other; then the radius is to the tangent of the difference between this angle and half a right angle, as the tangent of half the ſum of the angles at the baſe, to the tangent of half their difference.

Let ABC be a triangle of which the ſides AB, AC are unequal; and draw [a] AD at right angles to AB, and make it equal a 11. 1.
to AC, and join BD; therefore, BA is to AD or AC, as [b] the b 1. P. T.
radius to the tangent of the angle ABD. It is to be proved, that the radius is to the tangent of the difference between the angle ABD and the half of a right angle, as the tangent of half the ſum of the angles ACB, ABC to the tangent of half their difference.



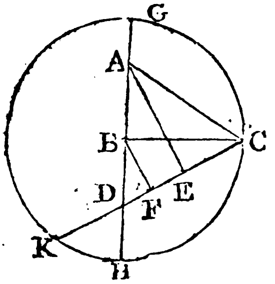
Make AE, AF each equal to AB, and join BE, BF, and draw [c] DG parallel to BE: c 31. 1.
and becauſe EA is equal to AB, the angle EBA is equal [d] to BEA; and EAB is a d 5. 1.
right angle; therefore each of the angles BEA, ABE is half a right angle [e]. For e 32. 1.
the ſame reaſon, each of the angles ABF, AFB is half of a right angle; therefore EBF is a right angle: and becauſe DG is parallel to BE; the angle DGE is a right angle [f], and each f 29. 1.
of the angles GDF, GFD half of a right angle; therefore DG is equal to GF: alſo, BG being the radius, DG is the tangent [b] of DBG, which is the difference between the angle DBA and FBA half a right angle: and DE is the ſum of the ſides CA, AB, and DF their difference: but becauſe DG is parallel to BF, BG is to GF or GD, as [g] ED to DF; that is, as the tan- g 2. 6.
gent of half the ſum of the angles ABC, ACB, to the tangent of half their difference [h]. Wherefore, &c. h 3. P. T.

Pl. Trig.

PROP. VII.

IN any triangle, as the rectangle contained by two ſides, is to the rectangle contained by half the ſum, and half the difference of the baſe and the difference of the ſides, ſo is the ſquare of the radius to the ſquare of the ſine of half the angle oppoſite to the baſe.

Let ABC be a triangle, of which the ſide AC is greater than AB; and make AD equal to AC; therefore BD is the difference of the ſides AC, AB: The rectangle BA, AC is to the rectangle contained by half the ſum, and half the difference of CB, BD, as the ſquare of the radius to the ſquare of the ſine of half the angle BAC.



a 12. 1. Join CD, and draw [a] AE, BF perpendicular to CD, and from
the centre B, at the diſtance BC, deſcribe a circle, meeting AD
in G, H, and CD in K: and becauſe
AD is equal to AC, the angle ACD
b 5. 1. is equal [b] to ADC, and AED, AEC
are right angles, and the ſide AE is
common to the triangles AED, AEC;
c 26. 1. therefore DE is equal [c] to EC, and
the angle DAE to EAC: and becauſe
BF drawn through the centre cuts CK
d 3. 3. at right angles, it biſects it [d]; therefore
CK is double of CF: and CD
is double of CE; therefore the remainder
DK is double of the remainder
EF: and becauſe the ſtraight lines CK, GH in the circle,
e 35. 3. cut one another, the rectangle CD, DK is equal [e] to the rectangle
GD, DH; and the rectangles contained by their halves
are equal; therefore the rectangle DE, EF is equal to the rectangle
contained by half of DG, the ſum of CB, BD, and half of
DH their difference: but becauſe BF is parallel to AE, DA is
f 2. 6. to AB, as DE to EF [f]; therefore the rectangles DA, AB, and
g 1. Def. 6. DE, EF, are ſimilar figures [g] upon AB, EF: and the ſquares of
AD, DE are ſimilar; therefore the rectangle DA, AB, or CA,
h 22. 6. AB, is the rectangle DE, EF, as [h] the ſquare of AD to the ſquare
of DE; that is, as the ſquare of the radius to the ſquare of the
k 1. P. T. ſine of DAE [k], half of BAC. Therefore the rectangle CA,
AB is to the rectangle contained by half the ſum, and half the
difference of CB, BD as the ſquare of the radius to the ſquare of
the ſine of half the angle BAC.

PROP.

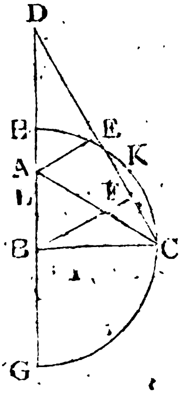
PROP. VIII.

Pl. Trig.

IN any triangle, the rectangle contained by two ſides, is to the rectangle contained by half the ſum of the three ſides, and its exceſs above the baſe, as the ſquare of the radius to the ſquare of the coſine of half the angle oppoſite to the baſe.

Let ABC be any triangle, the rectangle BA, AC is to the rectangle contained by half the ſum of the three ſides and its exceſs above the baſe BC, as the ſquare of the radius to the ſquare of the coſine of half the angle BAC.

In BA produced, take AD equal to AC; and join DC, and draw [a] AE, BF perpendicular to DC; and from the centre B, at the diſtance BC, deſcribe a circle, cutting DC again in K, and BD in H, G: and biſect [b] DG in L: and becauſe AD is equal to AC, the angles ADC, ACD are equal [c]: and AED, AEC are right angles; therefore DE is equal [d] to EC: and the angle BAC is equal [e] to ACD, ADC, it is therefore double of ADC: and becauſe BF, from the centre B, is at right angles to CK, it biſects it [f]; therefore CK is double of CF; and CD is double of CE; therefore the remainder DK is double of the remainder EF: For the ſame reaſon, DH is double of BL: and becauſe DG, DC cut the circle, the rectangle GD, DH is equal [g] to the rectangle CD, DK; and therefore the rectangles contained by their halves are equal; that is, the rectangle DE, EF is equal to the rectangle contained by DL half of DG the ſum of the three ſides, and BL the exceſs of LG that half ſum above BG or BC the baſe: but becauſe AE is parallel to BF, DA is to AB, as DE to EF [h]; therefore the rectangle DA, AB is ſimilar [k] to the rectangle DE, EF; and the ſquares of DA, DE are ſimilar; therefore the rectangle DA, AB, or CA, AB, is to the rectangle DE, EF, as the ſquare of DA to the ſquare of DE [l]; that is, as the ſquare of the radius to the ſquare of the coſine of ADC, half of the angle BAC [m]. Wherefore, as the rectangle contained by the ſides BA, AC is to the rectangle contained by half the ſum of the three ſides, and its exceſs above the baſe BC, ſo is the ſquare of the radius to the ſquare of the coſine of half the angle BAC.



a 12. 1.
b 10. 1.
c 5. 1.
d 26. 1.
e 32. 1.
f 3. 3.
g Cor. 36. 3.
h 16. 6.
k 1. Def. 6.
l 22. 6.
m 1. P. T.

Cor. Becauſe the rectangle contained by half the ſum of the three ſides, and its exceſs above the baſe, is to the rectangle BA, AC, as the ſquare of the coſine of half BAC to the ſquare of the radius [n]: and that the rectangle BA, AC is to the rectangle contained by half the ſum, and half the difference of the baſe and difference of the ſides, as the ſquare of the radius to the ſquare of the ſine of half BAC [o]; therefore, by equality [p], the rectangle contained by half the ſum of the three ſides, and its exceſs above the baſe, is to the rectangle contained by half the ſum, and half the difference of the baſe and difference of the ſides, as the ſquare of the coſine of half BAC to the ſquare of its ſine; that is, as the ſquare of the radius to the ſquare of the tangent [q] of half the angle BAC oppoſite to the baſe.

n 8. P. T.

o 7. P. T.

p 22. 6.

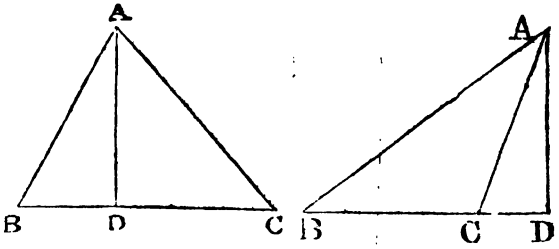
q 2. Cor. 10. Def.

SOLUTION of the CASES of OBLIQUE-ANGLED TRIANGLES.

GENERAL PROPOSITION.

In any triangle, of the three ſides, and any two of the angles, any three being given, the other two and the remaining angle may be found.

But if the three angles be given, the ratios only of the ſides are given, being the ſame with thoſe of the angles oppoſite to them: and in this caſe, the ſides cannot be found.



If two angles of a triangle be given, the third is alſo given, being the ſupplement of their ſum: and, on the contrary, if one of the angles be given, the ſum of the other two is alſo given.

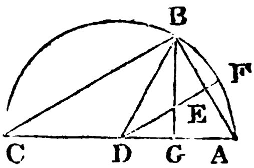
The four caſes of this propoſition may be reſolved by the help of ſome of the preceding propoſitions; as in the following Table.

Pl. Trig.

Caſes.	Given.	Sought.	Solution.
1	A, B and AB, two angles and a ſide.	BC AC	Sin. C : ſin. A : : AB : BC. Sin. C : ſin. B : : BA : AC. } P. 3.
2	AB, AC and B, two ſides, and an angle oppoſite to one of them.	The angles.	AC : AB : : ſin. B : ſin. C. P. 3. If AC be greater than AB, C is acute: otherwiſe it may be acute or obtuſe, by Cor. to Def. 8.
3	AB, AC and A, two ſides and the included angle.	The angles B & C and BC.	The ſum of BA, AC : diff. of BA, AC : : tan. of ½ ſum of B, C : tan. ½ diff. B, C. of P. 4. Otherwiſe, AB : AC : : R : tan. E; and R : tan. diff. of E and 45° : : tan. ½ ſum of B, C : tan. ½ diff. of B, C. P. 6. Whence B and C are found, by LEM. 4. Sin. B : ſin. A : : AC : BC.
4	AB, AC, BC, the three ſides.	A, B, C, the three angles.	Let AD be perp. to BC. BC : ſum of BA, AC : diff. of BA, AC : F; and BC, F together are double of the greater ſegment BD. P. 5. If BD be greater than CB, C is obtuſe; if not, it is acute: then AC : CD : : R : cos. C. Otherwiſe, Let D be the diff. of AB, AC, the rect. AB, AC : rect. ½ ſum BC, D and ½ diff. BC, D : : R^2 : ſin^2. ½ BAC. Otherwiſe, Let P be ½ ſum of the three ſides. The rect. BA, AC : rect. contained by P and the diff. of P, BC : : R^2 : cos^2. ½ BAC.

To conſtruct a Table of Sines, Tangents, &c.

Let ABC be a circle, of which the diameter is AC, and
the centre D: and let AB be any arch of it; and biſect AB in
a 30. 3. F [a]; and join AB, BC, BD, DF: and let DF meet AB in E;
b 12. 1. and draw [b] BG perpendicular to AC: and becauſe the arch
AB is double of AF, the angle ADB is
c 33. 6. double of ADF [c]: but ADB is double
d 20. 3. of ACB [d], therefore the angle ADF is
equal to ACB; and DE is parallel to
e 31. 3. BC, or at right angles to AB [e]; therefore
f 4. 6. AC is to CB, as AD to DE [f]; and
AC is double of AD; therefore CB is
g 14. 5. double of DE [g]: and becauſe the triangles
ADE, BCG are equiangular, AD is to DE, as BC to
CG [f]; therefore the rectangle AD, CG is equal to the rectangle
h 16. 6. BC, DE [h], that is, to double of the ſquare of DE [k]: and DE is
k 1. 2. the coſine of AF [l], and CG the ſum of the radius CD, and DG
l 1. Cor. the coſine of AB. Therefore, the rectangle contained by the
Def. 10. Pl. T. radius, and the ſum of the radius and the coſine of any arch,
is double of the ſquare of the coſine of half that arch.



If AB be equal to the radius AD, it is the ſide of a hexagon
m Cor. 15. 4. inſcribed in a circle [m]; and the arch AB is the ſixth part of the
circumference; that is, it is an arch of 60°; and becauſe then
n 47. 1. the triangle ABD is equilateral, AG is equal to GD [n]; that is,
it is the half of the radius AD. If, therefore, the radius be repreſented by unity, and the decimal notation be uſed, the ſum of the radius and the coſine DG is repreſented by 1.5, and the rectangle contained by the radius and this ſum is alſo repreſented by 1.5, becauſe the radius is 1; therefore half of this rectangle is .75, which is therefore the ſquare of DE the coſine of AF; that is, of 30°. Wherefore the coſine of 30° is the ſquare-root of .75; that is, .8660254039344+.

In the ſame manner, if the radius 1 be added to this number, half the ſum .9330127019672+ is the ſquare of the coſine of 15°; and therefore its ſquare-root .96592582632789 + is the coſine of 15°. In like manner, the ſquare-root of half the ſum of 1 and the coſine of 15° is .99144485936634 + the coſine of 7° 30′. In the ſame manner are found the coſine of 3° 45′, of 1° 57′ 30′, and ſo on: till at length, after twelve biſections of the arch of 60°, the coſine of $52'' \; 44''' \; 03'''' \; 45'''''$ is found to be .9999999673176965713. And if the ſquare of the coſine be ſubtracted from 1, which is the ſquare of the radius; the remainder is the ſquare of the ſine of the ſame arch [n]. Thus, the ſine of 52″ 44‴ 03⁗ 45′′′′′ is found to be .000255663461862138 32219. And becauſe the coſine of an arch is to its ſine, as the radius to the tangent of the ſame arch[o]; therefore the tangent of 52″ 44‴ 03⁗ 45′′′′′ is .00025566347, which is nearly equal to the ſine of it: but the arch is greater than the ſine, and leſs than the tangent; therefore the arch is more nearly equal to the ſine. Conſequently, in ſmall arches, the ſines are to one another as the arches. Wherefore, as 52″ 44‴ 03⁗ 45′′′′′ to 1′, ſo is .00025566 34618, the ſine of the firſt, to .0002908882086572, the ſine of 1′; and if the ſquare of this ſine be ſubtracted from 1, the ſquare-root of the remainder .9999999576920253 28 is the coſine of 1 minute. Pl. Trig. o 2. Cor. Def. P. T.

Again, becauſe the triangles ADE, ABG, have the angle at A common, and AED, AGB right angles, they are equiangular[p]; therefore AD is to DE, as AB to BG[f]: and if AF be an arch of 1′, AB is an arch of 2′; therefore, as the radius to the coſine of 1′, ſo is twice the ſine of 1′ to the ſine of two minutes. p 32. 1. f 4. 6.

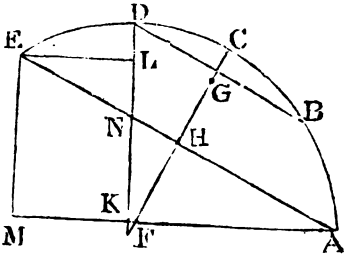
The ſines of all the other arches may be found by the following.

ID.

PROPOSITION

LET AB, AC, AD be three arches, ſuch that BC is equal to CD; as the radius to the coſine of DC the common difference, ſo is twice the ſine of AC the middle arch to the ſum of the ſines of AB, AD the extreme arches.

Make the arch DE equal to AB: and let F be the centre, and join FC, FD, AE, BD; and draw[a] AK, EL perpendicular to FD, and EM parallel[b] to it; therefore MK is equal[c] to EL the ſine of DE or AB: and AK is the ſine of AD[d]; therefore AM is the ſum of the ſines of AB, AD: and becauſe ED is equal to AB, and DC to CB, the whole EC is equal to CA; and therefore FC biſects AE, BD at right angles[e]; and AH, DG are the ſines of AC, CD; and FG is the coſine of DC: and becauſe the angle FDG is equal[f] to FNA, that is, to MEA; and FGD, EMA are right angles; the triangles FGD, AME are equiangular[g]; and therefore, as DF to FG, ſo is EA to AM[h]; that is, as the radius to the coſine of DC, ſo is twice AH the ſine of AC to the ſum of the ſines of AB, AD.



a 12. 1.
b 31. 1.
c 34. 1.
d 3. Def. P. T.
e Cor. 30. 3.
f 29. 1.
g 32. 1.
h 4. 6.

Let

Pl. Trig. Let DC, CB be arches of $1'$: then, if AC be $2'$, R is to cos. $1'$, as twice ſin. $2'$ to the ſum of the ſines of $3'$ and $1'$: If, therefore, a fourth proportional be found to R, cos. $1'$ and twice ſin. $2'$, the exceſs of this fourth proportional above ſin. $1'$ is the ſine of $3'$. In the ſame manner, if a fourth proportional be found to R, cos. $1'$ and twice ſin. $3'$, its exceſs above the ſin. $2'$ is the ſine of $4'$; and ſo on.

In this manner, the table of ſines may be computed: and from it the table of tangents may be calculated; for the coſine of any arch is to its ſine, as the radius to the tangent of the ſame arch [k]. As alſo the table of ſecants, becauſe the ſecant of any arch is a third proportional to its coſine and the radius [l].

k 2. Cor. Def. P. T.

l 3. Cor. Def. P. T.

To find the Length of the Circumference.

Becauſe the arch of $52''\ 44'''\ 03''''\ 45'''''$ is the 24576 the part of the circumference; and that the ſine is leſs than the arch; if the ſine of $52''\ 44'''\ 03''''\ 45'''''$ be multiplied by 24576, the product 6.28318524 is leſs than the circumference of the circle, of which the radius is 1: and becauſe the tangent is greater than the arch; if the tangent of $52''\ 44'''\ 03''''\ 45'''''$ be multiplied by 24576, the product 6.28318544 is greater than the circumference. Conſequently the circumference is nearly 6.28318531 when the radius is 1; or 3.14159265 when the diameter is 1.

Hence the diameter of a circle is to its circumference, nearly as 1 to 3.14159265; or as 7 to 22; or as 113 to 355.

THE

ELEMENTS

OF

SPHERICAL TRIGONOMETRY.

DEFINITIONS.

I.

A GREAT circle of the ſphere is the common ſection of the ſphere, with a plane paſſing through its centre. Sph. Trig.

II.

The pole of a great circle of the ſphere is a point on the ſuperficies of the ſphere, from which all ſtraight lines drawn to the circumference of the circle are equal.

III.

A ſpherical angle is that made by two arches of great circles, and is the ſame with the inclination of the planes of theſe circles.

IV.

A ſpherical triangle is a figure on the ſuperficies of the ſphere, contained by three arches of great circles, each of them leſs than a ſemicircle.

PROP. I.

IF a ſphere be cut by a plane paſſing through its centre, the common ſection is a circle.

For every ſtraight line drawn from the centre to the ſuperficies of the ſphere, is equal to the radius of the ſemicircle, by which the ſphere is deſcribed; therefore all ſtraight lines drawn from the centre of the ſphere to the common ſection of its ſuperficies with the cutting plane, are equal: and therefore the common ſection of the plane and the ſphere is a circle.

Cor. 1. The centre of a great circle is the ſame with the centre of the ſphere.

Cor. 2. All great circles are equal to one another; for their radii are equal to the radius of the circle, by which the ſphere is deſcribed.

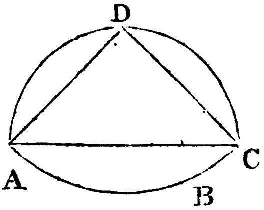
Cor. 3. Great circles biſect one another. For having the ſame centre, their common ſection is a diameter of each of them.

PROP. II.

THE arch of a great circle between the pole and the circumference of another great circle, is the fourth part of the circumference.

Let ABC be a great circle, and D its pole; and let the great circle ADC paſs through D, and meet the circumference ABC, in C: the arch DC is the fourth part of the circumference.

Let AC be the common ſection of
the circles ABC, ADC; therefore
a 3. Cor. 1. S. T. ADC is a ſemicircle [a]: join DA,
DC: and becauſe D is the pole of
b 2. Def. S. T. the great circle ABC, DA is equal [b]
to DC; therefore the arch AD is
c 28. 3. equal to the arch DC [c]; and the
arch DC is the half of ADC: but
the arch ADC is half of the circumference; therefore the arch DC is the fourth part of the circumference.



PROP. III.

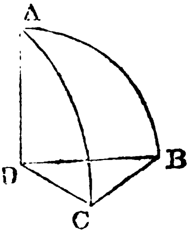
A Spherical angle made at the pole of a great circle, is meaſured by the arch of that great circle, intercepted between the circles, which contain the angle.

Let AB, AC be arches of great circles, which paſs through A the pole of the great circle BC, the arch BC is the meaſure of the ſpherical angle BAC.

Let the planes of the great circles cut one another in the ſtraight lines AD, DB, DC: and becauſe the great circles AB,
a 1. Def. S. T. AC paſs through the centre of the ſphere [a], their common

ſection AD paſſes through it. For the ſame reaſon, BD paſſes through it; therefore D is the centre of the ſphere: and becauſe A is the pole of the great circle BC, AB is the fourth part of the circumference [b]; therefore ADB is a right angle. For the ſame reaſon, ADC is a right angle; therefore the angle BDC is the inclination of the planes of the circles AB, AC [c]; it is therefore the ſame with the ſpherical angle BAC [d]: and the arch BC is the meaſure [e] of the angle BDC; therefore the arch BC is the meaſure of the ſpherical angle BAC.

b 2. S. T.
c 6. Def. 11.
d 3. Def. S. T.
e 1. Def. ET.



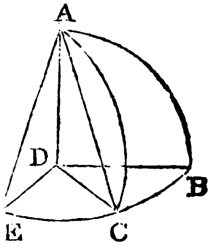
Cor. 1. The ſtraight line drawn from the pole A of a great circle BC to the centre of the ſphere D, is perpendicular to the plane of that great circle.

Cor. 2. The poles of a great circle are the extremities of the diameter of the ſphere, which is perpendicular to the plane of that great circle.

PROP. IV.

IF two arches of different great circles be drawn from the ſame point, and each of them be the fourth part of the circumference; that point is the pole of the great circle, which paſſes through the extremities of the arches.

Let AB, AC be arches of great circles not in the ſame plane, and each of them the fourth part of its circumference: The point A, in which they meet, is the pole of the great circle BCE which paſſes through their extremities.



Let the planes of the great circles cut one another in AD, DB, DC; therefore D is the centre of the ſphere. In the circumference BC take any point E; and join DE, AE, AC: and becauſe each of the arches AB, AC is the fourth part of the circumference, the angles ADB, ADC are right angles; therefore AD is perpendicular to the plane BDC [a]; and therefore ADE is a right angle: and AD, DE are equal to AD, DC, and ADE to ADC; therefore AE is equal to AC [b]; that is, any ſtraight line drawn from A to the circumference BC, is equal to AC; therefore A is the pole of BC [c].

a 4. 11.
b 4. 1.
c 2. Def. S. T.

Cor. 1. A great circle drawn through A, the pole of another great circle BC is perpendicular to it: for AD is perpendicular to the plane BDC[d]; therefore the plane ADB is perpendicular to it[e]; and ABC is a right angle[f].

d 1·Cor. 3. s. T.
e 18. 11.
f 3. Def. s. T.

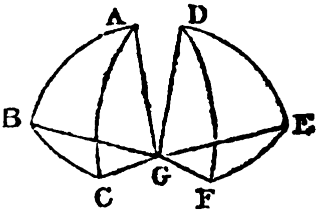
Cor. 2. If ABC be a right angle, the great circle AB paſſes through the pole of BC: For, if AB, BC be each of them the fourth part of the circumference, the arch AC is the meaſure of the ſpherical right angle ABC; it is therefore the fourth part of the circumference; and A is the pole of BC.

Cor. 3. If AB paſs through the pole of BC, BC ſhall paſs through the pole of AB: For ABC is a right angle.

PROP. V.

If two ſpherical triangles have the three ſides of the one equal to the three ſides of the other, each to each; the angles alſo, which are oppoſite to the equal ſides, ſhall be equal.

Let ABC, DEF be two ſpherical triangles, which have their ſides equal, AB to DE, AC to DF, and BC to EF; the angle BAC is equal to EDF, and ABC to DEF, and ACB to DFE.



Let G be the centre of the ſphere, and join AG, BG, CG, DG, EG, FG; theſe ſtraight lines are the common ſections of the planes of the great circles: and becauſe the arch AB is equal to DE, the angle AGB is equal[a] to DGE.

a 27. 3.

For the ſame reaſon, the angle AGC is equal to DGF, and BGC to EGF: and becauſe the plane angles AGB, AGC, BGC, which contain a ſolid angle at G, are equal to the angles DGE, DGF, EGF, which contain another ſolid angle at G; the planes in which the equal angles are, have the ſame inclination to one another[b]: and the ſpherical angle BAC is the inclination of the plane AGB to the plane AGC, and the ſpherical angle EDF the inclination of the planes DGE, DGF; therefore the ſpherical angle BAC is equal to EDF: and for the ſame reaſon, ABC is equal to DEF, and ACB to DFE.

b A. 11.

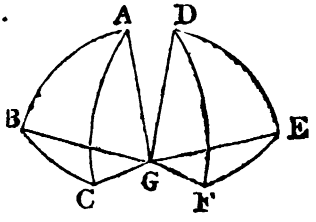
PROP.

PROP. VI.

Sph. Trig.

IF two ſpherical triangles have two ſides of the one equal to two ſides of the other, each to each; and have alſo the angles contained by thoſe ſides equal; their third ſides ſhall be equal; and the angles oppoſite to the equal ſides ſhall be equal, each to each.

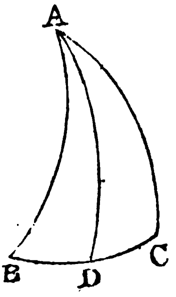
Let ABC, DEF be two ſpherical triangles, which have the ſides AB, AC equal to DE, DF, each to each, and the angle BAC equal to the angle EDF: the third ſides BC, EF ſhall be equal; and the angle ABC equal to DEF, and ACB to DFE.



For the ſame conſtruction being made, as in the preceding: becauſe AB, AC are equal to DE, DF, the angle AGB is equal [a] to DGE, a 27. 3. and AGC to DGF: and becauſe the ſpherical angle BAC is equal to EDF, the planes ABG, ACG have the ſame inclination with the planes EDG, FDG [b]: and becauſe b 3. Def. S. P. two of the plane angles AGB, AGC, BGC are equal to two of the plane angles DGE, DGF, EGF, and the planes in which they are have the ſame inclination to one another; the third angle BGC is equal [c] to the third angle EGF: therefore their c Cor. A. 11. meaſures BC, EF are equal; and therefore the ſpherical angle ABC is equal to DEF, and ACB to DFE, by the preceding.

PROP. VII.

THE angles at the baſe of an iſoſceles ſpherical triangle, are equal to one another.



Let ABC be an iſoſceles ſpherical triangle, having the ſide AB equal to AC; the angle ABC is equal to the angle ACB.

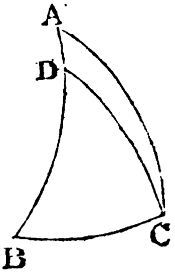
Biſect [a] the arch BC in D: and let a great a 30. 3. circle paſs through the points A, D: and becauſe the ſpherical triangles ABD, ACD have the ſides AB, BD equal to the two AC, CD, and the third ſide AD common to both; the angle ABD is equal [b] to the angle ACD. b 5. S. P.

PROP. VIII.

IF two angles of a ſpherical triangle be equal, the ſides oppoſite to them ſhall be equal.

Let ABC be a ſpherical triangle, of which the angle ABC is equal to ACB: the ſide AB is alſo equal to the ſide AC.

If not, let BD be equal to AC, and let a great circle paſs through the points D, C: and becauſe in the ſpherical triangles DBC, ACB, the two ſides DB, BC are equal to the two AC, CB, and the angle DBC to the angle a 6. s. P. ACB; the third ſides DC, AB are equal [a], and the angle DCB equal to ABC; that is, to ACB; the leſs to the greater; which is impoſſible: Therefore AB is not unequal to AC; that is, it is equal to it.

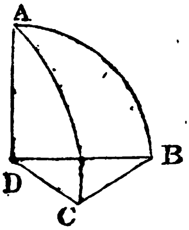


PROP. IX.

ANY two ſides of a ſpherical triangle are together greater than the third.

Let ABC be a ſpherical triangle: any two of its ſides are greater than the third.

Let D be the centre of the ſphere; and join DA, DB, DC; theſe ſtraight lines are the common ſections of the circles AB, AC, BC: and becauſe the ſolid angle at D is contained by the three plane angles ADB, ADC, BDC; any two of them are greater than the a 20. 11. third [a]: and the arches AB, AC, BC are the meaſures of theſe plane angles; therefore any two of the ſides AB, AC, BC are greater than the third.



PROP. X.

THE three ſides of a ſpherical triangle are together leſs than the circumference of a great circle.

Let ABC be a ſpherical triangle as before, the three ſides AB, AC, BC are together leſs than the circumference of a great circle.

Sph. Trig.

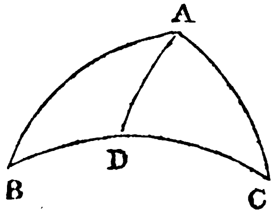
Let D be the centre of the ſphere; and join DA, DB, DC: and the ſolid angle at D is contained by the plane angles ADB, ADC, BDC; therefore theſe plane angles are leſs than four right angles: Wherefore alſo their meaſures AB, AC, BC are leſs than the meaſures of four right angles; that is, leſs than the whole circumference.

PROP. XI.

THE greater angle of a ſpherical triangle has the greater ſide oppoſite to it; and converſely.

Let ABC be a ſpherical triangle, having the angle BAC greater than the angle ABC; the ſide BC is alſo greater than the ſide AC.

Let the angle BAD be made equal to ABC; therefore AD is equal [a] to DB: to each of them add DC; and AD, DC are equal to BC: but AD, DC are greater [b] than AC; therefore BC is greater than AC.



[a] 8. s. p.

[b] 10. s. p.

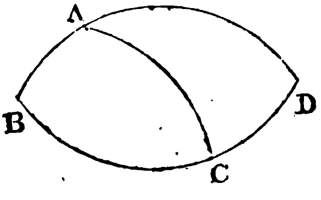
The converſe is demonſtrated, as Prop. XIX. 1. Elem.

PROP. XII.

IF two ſides of a ſpherical triangle be together greater than the half of the circumference of a great circle, the interior angles at the baſe are together greater than two right angles; and if equal, equal; and if leſs, leſs.

Let ABC be a ſpherical triangle, and let the ſides AB, AC together be greater than the half of the circumference of a great circle; the angles ABC, ACB together are greater than two right angles.

Produce BA, BC till they meet again in D: and becauſe great circles biſect one another [a], BAD is half of the circumference; therefore BA, AC are greater than BAD; take away BA, and AC is greater than AD; therefore the angle ADC is greater [b] than ACD: and ABC is equal to ADC, becauſe each of them



[a] 3. Cor. 1. s. p.

[b] 11. s. p.

is the inclination of the planes BAD, BCD; therefore ABC is greater than ACD: add ACB to each of them; and ABC, ACB are greater than ACD, ACB; that is, greater than two right angles.

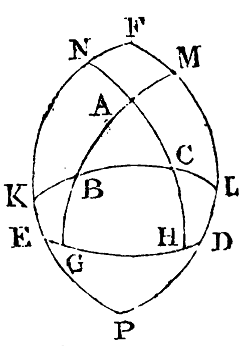
In the ſame manner, if BA, AC be equal to the half of the circumference; it may be proved, that the angles ABC, ACB are equal to two right angles: and if leſs, leſs.

COR. From the demonſtration, it is manifeſt, that if the ſides be together greater than the half of the circumference; the interior angle at the baſe is greater than the exterior and oppoſite angle; and if equal, equal; and if leſs, leſs.

PROP. XIII.

IF the angles of a ſpherical triangle be the poles of three great circles, the circumferences of theſe circles, by their interſections, ſhall form another triangle, ſuch that the ſides of one of the triangles are the ſupplements of the meaſures of the angles oppoſite to them in the other triangle: and therefore they are ſaid to be ſupplemental to one another.

Let ABC be a ſpherical triangle; and let A be the pole of the great circle DE, B the pole of DF, and C of EF; the ſides of the triangle DEF are the ſupplements of the meaſures of the angles at A, B, C, viz. DE of BAC, DF of ABC, and EF of ACB: alſo the ſide AB is the ſupplement of the meaſure of the angle EDF, AC of DEF, and BC of DFE.



Let AB produced, meet DE, DF in G, M, and AC meet DE, EF in H, N; and BC meet DF, FE in L, K: and becauſe the great circle AB paſſes through A the pole of DE, DE ſhall paſs through the pole of AB [a]. For the ſame reaſon, DF paſſes through the pole of AB; therefore their interſection D is the pole of AB. In the ſame manner, it may be proved, that E is the pole of AC, and F the pole of BC: and becauſe D is the pole of AB, the arch DG is the fourth part of the circumference [b]: and becauſe E is the pole of AC, EH is the fourth part of the circumference; therefore DG, EH, that is, DE, GH together, are half of the circumference; but GH is

a 3. Cor. 4. S. P.

b 2. S. P.

the meaſure of the angle BAC [c], becauſe A is the pole of GH; Sph. Trig. therefore DE is the ſupplement of the meaſure of the angle BAC. In the ſame manner, it may be proved, that DF is the c 3. s. p. ſupplement of the meaſure of ABC, and EF of ACB.

Again, becauſe A, B are the poles of DE, DF, each of the arches AG, BM is the fourth part of the circumference [b]; b 2. s. p. therefore AG, BM, that is, AB, GM together, are half of the circumference: and MG is the meaſure of the angle EDF [c], becauſe D is the pole of MG; therefore the meaſure of the angle EDF is the ſupplement of the ſide AB. In the ſame manner, it may be proved, that the meaſures of the angles DEF, DFE are the ſupplements of the ſides AC, BC, in the triangle ABC.

Cor. If the circles DF, EF meet again in P, the triangle DEP is ſemi-ſupplemental to the triangle ABC; that is, the ſide DE, and the meaſure of the oppoſite angle DPE, are the ſupplements of the meaſure of BAC, and of BC: but the ſides DP, PE are the meaſures of ABC, ACB; and the meaſures of PDE, PED are equal to the ſides AB, AC.

PROP. XIV.

THE three angles of a ſpherical triangle are together greater than two, and leſs than ſix right angles.

Becauſe the meaſure of each of the angles at A, B, C, together with the ſide oppoſite to it of the triangle DEF, is equal to half the circumference [a]; the meaſures of the three angles a 13. s. p. A, B, C, together with the ſides of the triangle DEF, are three halves of the circumference: and the three ſides of the triangle DEF are leſs than the circumference [b]; therefore the meaſures b 10. s. p. of the angles at A, B, C are together greater than half of the circumference; and the angles are therefore greater than two right angles.

And becauſe all the exterior and interior angles of a triangle are equal to ſix right angles, the interior angles are leſs than ſix right angles.

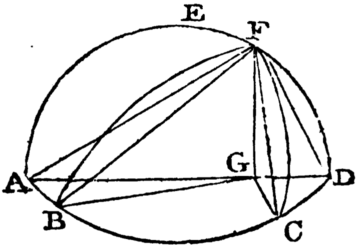
Cor. Becauſe DE, EF are greater than DF [c]; twice DF, to- c 9. s. p. gether with the meaſures of the angles at A, B, C, are leſs than three halves of the circumference [a]; and therefore the three angles at A, B, C, together with twice the ſupplement of the leaſt of them, are leſs than ſix right angles.

PROP.

PROP. XV.

IF arches of great circles be drawn to the circumference of any great circle, from a point in the ſuperficies of the ſphere, which is not its pole; the greateſt of them is that which paſſes through the pole, and its ſupplement is the leaſt; and of the others, that which is nearer to the greateſt, is greater than that which is more remote.

Let ABD be a great circle, of which E is the pole, and let F be any other point in the ſuperficies of the ſphere, and let the great circle AFD paſs through the points E, F, and meet the circle ABD in the diameter AD: Of all the arches of great circles FB, FC, &c. that can be drawn from F to the circumference ABD, FA is the greateſt, and FD the leaſt; and of the others, FB which is nearer to FA, is greater than FC which is more remote.



Draw FG perpendicular to AD, and join GB, GC, FA, FB, FC, FD: And becauſe the circle AFD is perpendicular to the circle ABD [a], for it paſſes through its pole E, and that FG is at right angles to their common ſection AD, FG is perpendicular [b] to the plane ABD; therefore FGB, FGC are right angles. And becauſe G is a point in the diameter AD, which is not the centre, for a ſtraight line from E to the centre is perpendicular to AD [c]; GA which paſſes through the centre is the greateſt [d], and GD the leaſt of all the ſtraight lines that can be drawn from G to the circumference ABD; and GB which is nearer to GA, is greater than GC which is more remote: and becauſe GA is greater than GB, the ſquares of AG, GF are greater than the ſquares of BG, GF; that is, the ſquare of AF is greater [e] than that of BF; therefore AF is greater than BF: and they are in equal circles; therefore the arch AF is greater [f] than the arch BF. In the ſame manner, it may be demonſtrated, that the arch BF is greater than the arch CF, and the arch CF than the arch DF. Wherefore FA is the greateſt, and FD the leaſt, and FB greater than FC.

[a] 1.Cor.4 s. p.

[b] 4.Def.11.

[c] 1.Cor.3. s. p.

[d] 7. 3.

[e] 47. 1.

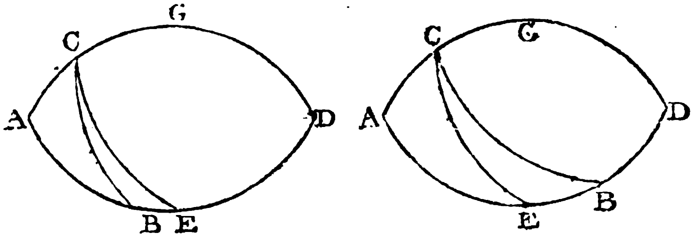
[f] 15. 3.

PROP. XVI.

IN a right angled ſpherical triangle, the ſides about the right angle are of the ſame affection with their oppoſite angles; that is, if one of the ſides be greater than the fourth part of the circumference, the angle oppoſite to it is greater than a right angle; and if equal, equal; and if leſs, leſs.

Let ABC be a ſpherical triangle, having BAC a right angle, the ſide AB is of the ſame affection with the oppoſite angle ACB.

If AB be the fourth part of the circumference, B is the pole [a] of AC, and ACB is a right angle [b]. a 2.Cor.4. S. P. b 1.Cor.4. S. P.



If AB be not the fourth part of the circumference, take AE the fourth part of it, and let a great circle paſs through E and C; therefore E is the pole of AC [a], and ECA is a right angle [b]: But if AB be greater than AE, the angle ACB is greater than ACE; and if AB be leſs than AE, the angle ACB is leſs than ACE; that is, than a right angle. a 2.Cor.4. S. P.

PROP. XVII.

IN a right angled ſpherical triangle, if the ſides about the right angle be of the ſame affection, the hypotenuſe is leſs than the fourth part of the circumference; and if they be of different affection, the hypotenuſe is greater than the fourth part of the circumference: or if one of the other two be the fourth part of the circumference, the hypotenuſe is equal to it.

Let ABC be a ſpherical triangle, having the right angle BAC; if one of the ſides AB, AC be the fourth of the circumference, the hypotenuſe BC is alſo the fourth; but if AB, AC be both leſs, or both greater, the hypotenuſe is leſs than the fourth part of the circumference; and if one of them be leſs, and the other greater, the hypotenuſe is greater.

Let the circumferences AB, AC meet again in D, and biſect the arch ABD in E; therefore E is the pole of ACD [a]; biſect alſo ACD in G, and G is the pole of ABD, and let a great circle paſs through E, C: If the point B coincide with E, that is, if AB be the fourth part of the circumference, BC coincides with EC, and is the fourth part of the circumference. If AB, AC be both leſs than AE or AG; CE is nearer to CGD which paſſes through the pole G, than CB is; therefore CB is leſs [b] than CE the fourth part of the circumference. In the ſame manner, in the triangle CBD, of which the ſides CD, BD are each of them greater than CE, the hypotenuſe CB is leſs than CE.

a 2. s. p.

b 15. s. p.

But if AB be greater than AE, and AC leſs, the arch CB is nearer to CGD than CE is; therefore CB is greater [b] than CE the fourth part of the circumference.

Cor. 1. On the contrary, if the hypotenuſe be leſs than the fourth part of the circumference, the ſides are of the ſame affection; for if they be of different affection, the hypotenuſe is greater than the fourth: and if the hypotenuſe be greater than the fourth part of the circumference, the ſides are of different affection.

Cor. 2. Hence, in a right angled triangle, if the angles be of the ſame affection, the hypotenuſe is leſs than the fourth part of the circumference: and if they be of different affection, the hypotenuſe is greater than the fourth: and converſely. For the angles are of the ſame affection with their oppoſite ſides.

PROP. XVIII.

In any ſpherical triangle, if the perpendicular drawn from the vertex to the baſe, fall within the triangle, the angles at the baſe are of the ſame affection: and if it fall without, they are of different affection.

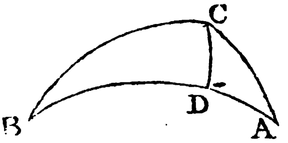
Let ABC be a ſpherical triangle, and let the arch CD of a great circle be drawn from C perpendicular to AB.

First,

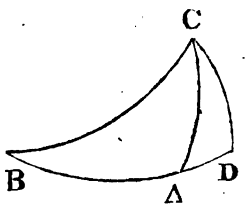
Sph. Trig.

Firſt, Let CD fall within the triangle ABC; and becauſe ADC is a right angled triangle, the angle CAD is of the ſame affection with the oppoſite ſide CD [a]. For the ſame reaſon, the angle CBA is of the ſame affection with CD; therefore the angles CAB, CBA are of the ſame affection.

a 16. s. T.



Next, Let the perpendicular CD fall without the triangle; and becauſe CDA is a right angle, the angle CAD is of the ſame affection with CD [a]; that is, with the angle CBD; but CAD, CAB are of different affection, becauſe they are together equal to two right angles; therefore the angles CAB, CBA are of different affection.



Cor. Hence, if the angles at A, B be of the ſame affection, the perpendicular falls within the triangle; for, if it fell without, they would be of different affection. And if the angles at A, B be of different affection, the perpendicular falls without the triangle; for, if it fall within, they would be of the ſame affection.

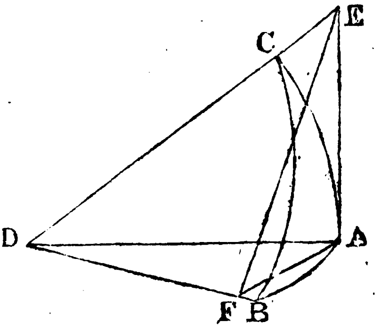
PROP. XIX.

IN a right angled ſpherical triangle, the ſine of either of the ſides about the right angle, is to the radius of the ſphere, as the tangent of the other ſide to the tangent of the angle oppoſite to it.

Let ABC be a ſpherical triangle, having the right angle BAC; as the ſine of the ſide AB to the radius, ſo is the tangent of the ſide AC to the tangent of the angle ABC.

Let D be the centre of the ſphere, and join DA, DB, DC, and from the point A draw AF perpendicular [a] to BD; and from F draw [b] FE, in the plane BDC, at right angles to BD; and let it meet DC in E, and join AE: and becauſe DF is at right angles, both to AF and FE, it is perpendicular [c] to the plane AFE;

a 12. 1.
b 11. 1.
c 4. 11.



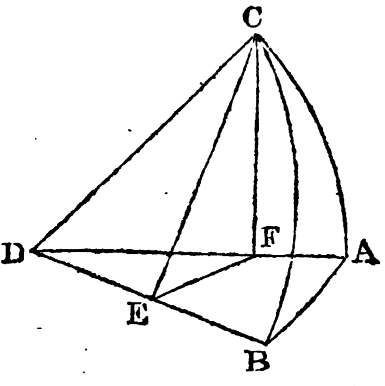
therefore alſo the plane ADB paſſing through DF, is perpendicular [d] to the plane AFE; and AFE is perpendicular to ADB: but the plane ADC or ADE is alſo perpendicular to ADB, becauſe BAC is a right angle; therefore their common ſection AE is perpendicular [e] to the plane ADB; and the angles EAD, EAF are therefore right angles; wherefore AE is the tangent of the arch AC: and becauſe, in the plane triangle AFE, EAF is a right angle; the ſide AF is to the radius, as AE to the tangent of the angle AFE: but AF is the ſine of AB, and AE the tangent of AC, and the angle AFE is the inclination of the planes ADB, BDC, for AF, FE are at right angles to their common ſection DB; it is therefore the ſame with the ſpherical angle ABC: Therefore the ſine of the ſide AB is to the radius, as the tangent of the ſide AC to the tangent of the oppoſite angle ABC.

d 18. 11.
e 19. 11.

PROP. XX.

IN a right angled ſpherical triangle, the ſine of the hypotenuſe is to the radius, as the ſine of either ſide to the ſine of the angle oppoſite to it.

Let ABC be a ſpherical triangle, have the right angle BAC; the ſine of the hypotenuſe BC is to the radius, as the ſine of the ſide AC to the ſine of the angle ABC.



Let D be the centre of the ſphere, and join DA, DB, DC; and from C draw [a] CE perpendicular to BD; and from E draw [b] EF, in the plane ADB, at right angles to DB, and join CF: and it may be demonſtrated, as in the preceding propoſition, that CFD, CFE are right angles: and becauſe in the plane triangle CEF, CFE is a right angle; therefore, as CE is to the radius, ſo is CF to the ſine of the angle CEF: but CE is the ſine of the arch CB, and CF the ſine of CA, and the angle CEF is the inclination of the planes ADB, CDB; that is, it is the ſame with the ſpherical angle ABC: Therefore the ſine of the hypotenuſe CB is to the radius, as the ſine of the ſide CA to the ſine of the oppoſite angle ABC.

a 12. 1.
b 11. 1.

OF THE GENERAL PROPORTIONS.

IN a right angled ſpherical triangle, if the complements of the ſides about the right angle be taken inſtead of the ſides themſelves; and theſe complements be ſuppoſed to be joined together at the right angle, but to be ſeparated from the hypotenuſe by the other two angles coming between them; and theſe five be called parts of the triangle: all the caſes of right angled triangles may be reſolved by two general proportions deduced from the two preceding propoſitions.

PROPORTION I.

As the radius to the coſine of any part of a right angled ſpherical triangle, ſo is the tangent of either of the parts adjacent to it, to the cotangent of the other adjacent part.

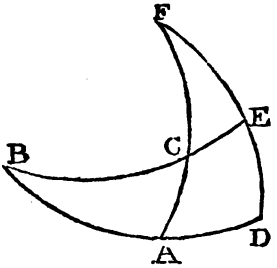
PROPORTION II.

As the radius to the ſine of any part of a right angled ſpherical triangle, ſo is the ſine of either of the parts adjacent to it, to the coſine of the part ſeparated from them both.

Firſt, Let the complement of AB one of the ſides about the right angle BAC be taken, with which to compare the other parts; then the angle at B and the complement of AC are adjacent to it: and by Prop. XIX. as the radius is to the ſine of AB, that is, to the coſine of its complement, ſo is the tangent of the angle at B to the tangent of AC, or the cotangent of its complement, which is the ſame with the firſt general proportion. Alſo the complement of AB is ſeparated from BC and the angle at C: and Prop. XX. the radius is to the ſine of BC, as the ſine of the angle at C to the ſine of AB, that is, to the coſine of its complement; which is the ſame with the ſecond general proportion.

Secondly, Let ABC one of the angles be taken, with which to compare the other parts; then BC and the complement of AB are adjacent to it.

Let the great circle DEF be deſcribed, of which B is the pole; and let it meet the ſides of the triangle in the points D, E, F; therefore the angles at D, E are right angles [a]: and becauſe the angles at A, D are right angles, each of the circles AC, DE paſſes through the pole of BD [b]; their interſection F is therefore the pole of BD; and therefore each of the arches FD, FA, BD, BE is the fourth part of the circumference: But in the triangle FCE right angled at E, by Prop. XIX. the radius is to the ſine of EF, as the tangent of F to the tangent of CE; that is, as the cotangent of CE to the cotangent of F [c], becauſe the tangents of two arches are reciprocally proportional to their cotangents [c]: and FE is the complement of ED the meaſure of the angle at B, and CE is the complement of CB, and the angle at F is meaſured by AD the complement of AB; therefore as the radius to the coſine of B, ſo is the tangent of BC to the cotangent of the complement of AB; and this is the ſame with the firſt proportion.



Likewiſe, the angle ABC is ſeparated from the angle at C and the complement of AC: and in the triangle CEF, by Prop. XX. the radius is to the ſine of CF, as the ſine of the angle at C to the ſine of EF: and CF is the complement of CA, and EF the complement of ED or ABC; therefore the radius is to the ſine of the complement of AC, as the ſine of the angle at C to the coſine of ABC; which is the ſame with the ſecond proportion.

Thirdly, Let the hypotenuſe BC be taken, with which to compare the other parts; then the angles at B, C are adjacent to it: and in the triangle CEF, by Prop. XIX. the radius is to the ſine of CE, as the tangent of C to the tangent of EF: and CE, EF are the complements of BC, and B; therefore the radius is to the coſine of BC, as the tangent of C to the cotangent of B, which is the firſt proportion.

Alſo, BC is ſeparated from the complements of AB and AC: and in the triangle CEF, by Prop. XX. the radius is to the ſine of CF, as the ſine of F to the ſine of CE: and CF and F are the complements of CA and AB, and CE of CB; therefore the radius is to the ſine of the complement of AC, as the ſine of the complement of AB to the coſine of BC; which is the ſecond general proportion.

[a] 1. Cor. 4. S. T.

[b] 2. Cor. 4. S. T.

[c] 5. Cor. 10. Def. P. T.

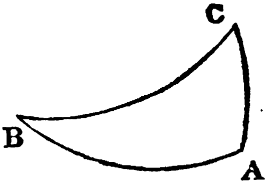
SOLUTION of the CASES of RIGHT-ANGLED SPHERICAL TRIANGLES.

PROBLEM.

Sph. Trig.

IN a right angled triangle, of the three ſides and three angles, any two being given, beſides the right angle, to find the other three.

This problem has ſixteen caſes, the ſolutions of which, deduced from the preceding general proportions, are contained in the following table, where ABC is any ſpherical triangle having the right angle BAC.



Given.

Sph. Trig.

Given.	Sought.	Solution.	G. Proportion.	Limitation.	Caſes.
BC & B	AC	R : ſin. BC : : ſin. B : ſin. AC.	2	of the ſame affection with B, (16).	1
	AB	R : cos. B : : tan. BC : tan. BA.	1	leſs than 90°, when BC and B are of the ſame affection.	2
	C	R : cos. BC : : tan. B : cot. C.	1	otherwiſe greater than 90°.	3
AC & C	AB	R : ſin. AC : : tan. C : tan. AB.	1	of the ſame affection with C, (16).	4
	BC	Cos. C : R : : tan. AC : tan. BC.	1	leſs than 90° when AC and C are of the ſame affection.	5
	B	R : ſin. C : cos. AC : cos. B.	2	of the ſame affection with AC, (16).	6
AC & B	AB	tan. B : tan. AC : R : ſin. AB.	1	ambiguous ; for two triangles may have the given	7
	BC	ſin. B : R : : ſin. AC : ſin. BC.	2	things, but have the things ſought in one of them	8
	C	cos. AC : R : : cos. B : ſin. C.	2	the ſupplements of the things ſought in the other.	9
AC & CB	AB	cos. AC : R : : cos. BC : cos. BA.	2	leſs than 90°, if AC and CB be of the ſame affection.	10
	B	ſin. BC : R : : ſin. AC : ſin. B.	2	of the ſame affection with AC, (16).	11
	C	tan. CB : tan. CA : : R : cos. C.	1	leſs than 90°, if AC and CB be of the ſame affection.	12
AB & AC	BC	R : cos. AC : : cos. AB : cos. BC.	2	leſs than 90°, if AB and AC be of the ſame affection, (17)	13
	B	ſin. AB : R : : tan. AC : tan. B.	1	of the ſame affection with AC, (16).	14
	C	ſin. AC : R : : tan. AB : tan. C.	1	of the ſame affection with AB, (16).	14
B & C	AB	ſin. B : R : : cos. C : cos. AB.	2	of the ſame affection with C, (16).	15
	AC	ſin. C : R : : cos. B : cos. AC.	2	of the ſame affection with B, (16).	15
	BC	tan. B : cot. C : : R : cos. BC.	1	leſs than 90°, if B and C be of the ſame affection, (2. Cor. 17).	16

All the limitations are either contained in the 16th and 17th propoſitions, or elſe they are eaſily deducible from them. Thus, becauſe C is of the ſame affection with AB; if AC and C be of the ſame affection, AC and AB are of the ſame affection; and therefore BC is leſs than 90°, (Prop. XVII.); this is the limitation in caſe 5. Again, if BC be leſs than 90°, AC and AB, or B and C, are of the ſame affection; and therefore, if AC be leſs than 90°, or of the ſame affection with CB, AB is leſs than 90°; and for the ſame reaſon, C is leſs than 90°. But if AC, BC be of different affection, that is, AC greater than 90°, BA is greater than 90°: and the ſame may be proved, in the ſame manner, when BC is greater than 90°. Thus, the reaſon of the limitation in the 2d, 3d, 10th, and 12th caſes, is manifeſt.

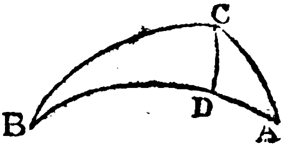
PROP. XXI.

IN any ſpherical triangle, the ſines of the ſides are proportional to the ſines of their oppoſite angles.

Let ABC be a ſpherical triangle, as the ſine of AC to the ſine of BC, ſo is the ſine of the angle at B to the ſine of the angle at A.

If one of the angles, as ACB be a right angle; the ſine of AC is to the ſine of B as [a] the ſine of the hypotenuſe AB to the radius or ſine of the right angle ACB; that is, as the ſine of BC to the ſine of A.

But if none of them be a right angle, let CD be the arch of a great circle, perpendicular to AB: and becauſe ADC is a right angle, the ſine of AC is to the radius, as the ſine of CD to the ſine of A [a]. For the ſame reaſon, the radius is to the ſine of CB, as the ſine of B to the ſine of CD; therefore, by perturbate equality [b], the ſine of AC is to the ſine of CB, as the ſine of B to the ſine of A.



IN oblique angled triangles, if a perpendicular be drawn from the vertex to the baſe, or the baſe produced; and the ſegments of the baſe, or the ſegments of the vertical angle, contained between the perpendicular and the ſides, be taken, with which to compare the other parts of the triangle: moſt of the caſes of oblique angled triangles may be reſolved by the following general proportion.

[a] 10. S. T.

[b] 23. 5.

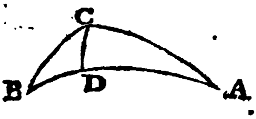
GENERAL PROPORTION III.

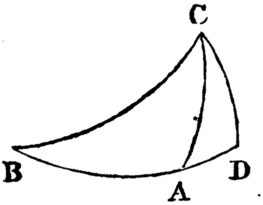
The ſines of the parts adjacent to the perpendicular, are reciprocally proportional to the tangents of the parts adjacent to them; and directly proportional to the coſines of the parts ſeparated from them.

But the coſines of the parts adjacent to the perpendicular are to be uſed inſtead of their ſines, when they are compared with the ſides of the triangle.

Let ABC be an oblique angled triangle, and let CD be perpendicular to AB, and fall either within or without the triangle.

Firſt, Let the ſegments of the baſe AD, DB be taken, with which to compare the other parts of the triangle ABC; then the angles at A, B are the parts adjacent to them: and becauſe ADC is a right angled triangle, the ſine of AD is to the radius,
a 19. S. T. as the tangent of CD to the tangent of A [a]. For the ſame reaſon, the radius is to the ſine of BD, as the tangent of B to the tan-
b 23. 5. gent of CD; therefore, by perturbate equality [b], the ſine of AD is to the ſine of BD, as the tangent of B to the tangent of A; which is the firſt part of the proportion.





Alſo, the ſides AC, CB are ſeparated from AD and DB: and becauſe ADC is a right angle, the coſine of AD is to the radius,
c 2. G. P. as the coſine of AC to the coſine of CD [c]: and for the ſame reaſon, the radius is to the coſine of DB, as the coſine of DC to
d 22. 5. the coſine of CB; therefore, by equality [d], as the coſine of AD to the coſine of DB, ſo is the coſine of AC to the coſine of CB; which is the ſecond part of the general proportion.

Secondly, Let the vertical angles ACD, BCD be taken, with which to compare the other parts of the triangle ABC; then, the ſides AC, CB are adjacent to the vertical angles: and becauſe ADC is a right angle, the coſine of ACD is to the radius, as the tangent of CD to the tangent of CA [e]: and for the ſame reaſon, the radius is to the coſine of BCD, as the tangent of BC to the tangent of CD; therefore, by perturbate equality [b], as the coſine of ACD to the coſine of BCD, ſo is the tangent of BC to the tangent of CA; which is the firſt part of the proportion. e 1. G. P. b 23. 5.

Alſo, the angles at A, B are ſeparated from the vertical angles ACD, BCD: and becauſe ADC is a right angle, the ſine of ACD is to the radius, as the coſine of A to the coſine of CD [c]: and for the ſame reaſon, the radius is to the ſine of BCD, as the coſine of CD to the coſine of B; therefore, by equality [d], as the ſine of ACD to the ſine of BCD, ſo is the coſine of A to the coſine of B; which is the ſecond part of the general proportion. c 2. G. P. d 22. 5.

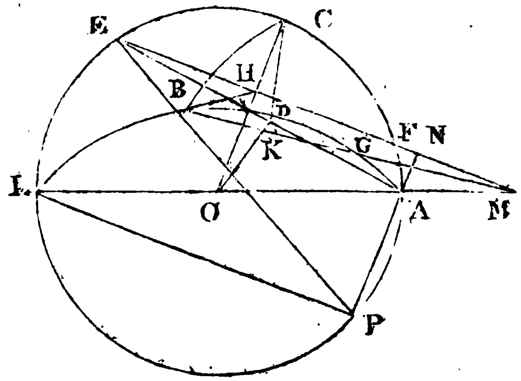
PROP. XXII.

IN any ſpherical triangle, if a great circle be drawn from the vertex, perpendicular to the baſe; the tangent of half the ſum of the ſegments of the baſe, is to the tangent of half the ſum of the ſides, as the tangent of half the difference of the ſides to the tangent of half the difference of the ſegments of the baſe.

Let ABC be a ſpherical triangle, and let the arch CD be perpendicular to AB; the tangent of half the ſum of AD, DB is to the tangent of half the ſum of the ſides AC, CB, as the tangent of half the difference of AC, CB, to the tangent of half the difference of AD, DB.

Let AL be the common ſection of the planes AC, AB, and O the centre; make CE, CF each equal to CB; therefore AE is the ſum of AC, CB, and AF is their difference: and if AC be equal to CL, A is the pole of CD; therefore AD is equal to DL; and the tangent of half the difference of AC, CB is the cotangent of half their ſum, and the tangent of half the difference of AD, DB is the cotangent of half their ſum. But if AC be not equal to CL, join EF, and let it meet AL in M; and join BM meeting the arch AB again in G; and join CO, OD, and let them meet EF, BG in H, K; and join BH; and through A draw [a] AN parallel to OC, meeting the circle ACL again in P; and join EA, EP, PL: and becauſe EC is equal to CF, OC biſects EF at right angles in H [b]: and becauſe CB is equal to CE, CH is the verſed ſine of CB, and therefore OHB is a right angle [c]: and OHE is a right angle; therefore OH is a 31. 1. bCor.30.3. c 4. Def. P. T.

perpendicular to the plane MBH [d]; the plane MBH is therefore
perpendicular [e] to the plane COD paſſing through OH: and the
d 4. 11. plane ADB is alſo perpendicular to the plane COD, becauſe
e 18. 11. ADC is a right angle; therefore their common ſection BG is
f 19. 11. perpendicular [f] to the plane COD; the angle OKB is therefore
g 3. 3. a right angle, and BK is equal to KG [g], and the arch BD to the
h 30. 3. arch DG [h]; therefore AB, AG are the ſum and difference of
AD, DB: and becauſe AN is parallel to OC, ANE is a right
k 29. 1. angle [k]; therefore, if EN be made the radius, AN is the tan-
l 1. P. T. gent [l] of AEN, half of the angle at the centre ſtanding upon
m 20. 3. AE [m], and PN is the tangent of PEN, or the cotangent [n] of
n 10. Def. P. T. EPN, that is, of half the arch AE: but becauſe the angle
o 31. 3. APL in the ſemicircle is equal [o] to the right angle ANM; AM is
p 2. 6. to ML, as AN to NP [p]; that is, as the tangent of half AF to
the cotangent of half AE. In the ſame manner, if a ſtraight



line be drawn through A, parallel to OD, it may be proved, that
AM is to ML, as the tangent of half AG to the cotangent of half
AB; therefore, as the tangent of half AF to the cotangent of
q 11. 5. half AE, ſo [q] is the tangent of half AG to the cotangent of
r 16. 5. half AB; and, alternately [r], the tangent of half AG is to the
tangent of half AF, as the cotangent of half AB to the cotan-
gent of half AE; that is, as the tangent of half AE to the
tangent of half AB, becauſe the tangents of two arches are re-
s 5. Cor. 10. ciprocally proportional to their cotangents [s]. Wherefore, &c.
Def. P. T. Cor. Hence, the tangent of half the baſe is to the tangent of
half the ſum of the ſides, as the tangent of half the difference
of the ſides to the tangent of half the difference of the ſegments
of the baſe, when the perpendicular falls within, or to the tan-
gent of half the ſum of the ſegments when it falls without the
triangle.

LEMMA.

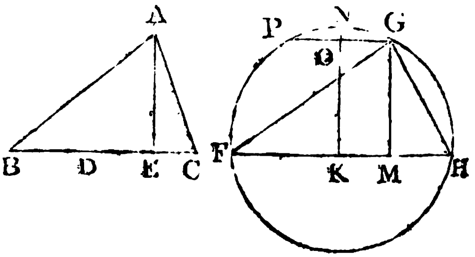
LEMMA.

Sph. Trig.

IN a plane triangle, the rectangle contained by half of the baſe and the diſtance of the perpendicular from the middle of the baſe, is equal to the rectangle contained by the ſines of half the ſum and half the difference of the arches of any circle, which are ſubtended by the ſides of the triangle.

Let ABC be a triangle, and biſect the baſe BC in D, and draw AE perpendicular to BC: and let FGH be any circle in which are placed FG equal to AB, and GH to AC: the rectangle BD, DE is equal to the rectangle contained by the ſines of half the ſum and half the difference of the arches FG, GH.

Join FH, and biſect [a] it in K; and draw GM, KN perpendicular to KH [b], and GOP parallel to it [c]; therefore the angle FGP is equal to GFH [d]; and the arch FP equal to GH [e]: and becauſe KN biſects FH at right angles, it paſſes through the centre [f], and therefore it biſects [g] the ſtraight line GP and the arches FGH, PNG [h]. Wherefore FK is the ſine [k] of FN half the ſum of the arches FG, GH, and KM or GO is the ſine [k] of half PG the difference of the arches FG, GH: but four times the rectangle FK, KM is equal to twice the rectangle FH, KM; that is, to the difference of the ſquares of FG, GH [l], or of BA, AC; and four times the rectangle BD, DE is alſo equal to the difference of the ſquares of BA, AC [l]; therefore the rectangle BD, DE is equal to the rectangle FK, KM, contained by the ſine of half the ſum of the arches FG, GH and the ſine of half their difference.



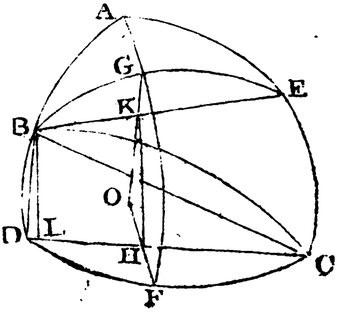
a 10. 1.
b 11. 12. 1.
c 31. 1.
d 29. 1.
e 27. 3.
f Cor. 1. 3.
g 3. 3.
h 30. 3.
k 3. Def. P. T.
l C. 2.

PROP. XXIII.

IN any ſpherical triangle, as the rectangle contained by the ſines of two ſides, is to the rectangle contained by the ſines of half the ſum and half the difference of the baſe and the exceſs of the ſides, ſo is the ſquare of the radius to the ſquare of the ſine of half the angle oppoſite to the baſe.

Let ABC be a ſpherical triangle, of which the ſide AC is greater than AB; and make AD equal to AC, and AE to AB; the rectangle contained by the ſines of BA, AC is to the rectangle contained by the ſine of half the ſum of CB, BD, and the ſine of half the difference of the ſame CB, BD, as the ſquare of the radius to the ſquare of the ſine of half the angle BAC.

Deſcribe through C, D and B, E the great circles CFD,
BGE, and biſect the angle BAC by the great circle AGF, and
join BC, BD, DC, BE; and let O be the centre of the ſphere,
and join OF, OG, meeting CD, BE in H, K, and join HK, and
a 11. 1. draw [a] BL perpendicular to CD. And becauſe in the ſpherical
triangles CAF, DAF, the two ſides CA, AF, are equal to the
two DA, AF, and the angle CAF
is equal to DAF; therefore the
b 6. s. T. baſe CF is equal to DF [b], and
the angle CFA to AFD; therefore
the plane AGF is perpen-
c 3. Def. dicular [c] to the plane CFD: and
s. T. CD is perpendicular to their
common ſection OF, becauſe OF,
which biſects the arch CFD,
dCor. 30. 3. biſects CD at right angles [d];
e 4. Def. 11. therefore CD is perpendicular [e]
to the plane AGF, and DHK is
therefore a right angle: For the
ſame reaſon, BE is perpendicular to the plane AGF; therefore
f 6. 11. BK is parallel to DH [f]: and BL, HK are parallel, becauſe BLH,
LHK are right angles; therefore BL HK is a parallelogram,
h 34. 1. and BK is equal to LH [h]: and becauſe the ſides CB, BD of the
triangle CBD are placed in equal circles CB, ABD, the rect-
angle CH, HL, or CH, BK, is equal to the rectangle contained
k Lem. by half the ſum and half the difference of the arches CB, BD [k];
s. T. but in the right angled ſpherical triangle AGB, the ſine of AB
l 20. s. T. is to BK the ſine of BG, as the radius to the ſine of BAG [l]
half of BAC; and in the right angled triangle ACF, the ſine
of



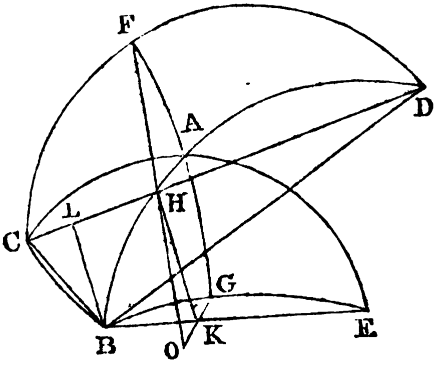
of AC is to CH the ſine of CF, as the radius to the ſine of FAC *Sph. Trig.* half of BAC [l]; that is, as the ſine of AB to BK; therefore the rectangle CH, BK is ſimilar to the rectangle contained by the ſines of AB, AC: Wherefore, as the rectangle contained by the ſines of BA, AC is to the rectangle CH, BK contained by the ſines of half the ſum and half the difference of the arches CB, BD, ſo is the ſquare of the radius to the ſquare of the ſine of half the angle BAC [m]. m 22. 6.

PROP. XXIV.

IN any ſpherical triangle, the rectangle contained by the ſines of two ſides, is to the rectangle contained by the ſine of half the ſum of the three ſides, and the ſine of its exceſs above the baſe, as the ſquare of the radius to the ſquare of the coſine of half the angle oppoſite to the baſe.

Let ABC be a ſpherical triangle; the rectangle contained by the ſines of two ſides AB, AC is to the rectangle contained by the ſine of half the ſum of the three ſides, and the ſine of the exceſs of the ſaid half ſum above the baſe BC, as the ſquare of the radius to the ſquare of the coſine of half the angle BAC.

Produce BA, CA to D, E, and make AD equal to AC, and AE to AB; and deſcribe the great circles CFD, BGE; and let the great circle GAF biſect the angle CAD, and meet the great circles CFD, BGE in F, G; and join EB, BC, CD, DB; and draw BL perpendicular to DC; and let O be the centre of the ſphere, and join OF, OG, meeting CD, BE, in H, K; and join HK.



It may be demonſtrated, as in the preceding propoſition, that HL is equal to BK; and that the rectangle contained by the ſines of BA, AC is to the rectangle BK, CH, or CH, HL, as the ſquare of the radius to the ſquare of the ſine of half the angle BAE or CAD: but, becauſe CD is biſected in H, the rectangle CH, HL is equal to the rectangle contained by the ſines of half the ſum, and half the difference of the arches BD, BC, and the ſum of BD, BC is the ſum of the three ſides, and the exceſs of the half of BD above half of BC is the exceſs of half the ſum of the three ſides above BC; therefore the rectangle CH, HL is equal to the rectangle contained by the ſines of half the ſum of the three ſides, and of its exceſs above BC: alſo, becauſe the angles CAB, CAD are equal to two right angles, the half of CAB and the half of CAD are together equal to a right angle; therefore the ſine of half CAD is the coſine of half CAB: But the rectangle contained by the ſines of BA, AC, is to the rectangle CH, HL as the ſquare of the radius to the ſquare of the ſine of half the angle CAD; therefore the rectangle contained by the ſines of BA, AC, is to the rectangle contained by the ſines of half the ſum of the three ſides, and of its exceſs above the baſe BC, as the ſquare of the radius to the ſquare of the coſine of half the angle BAC.

Cor. Becauſe the rectangle contained by the ſines of half the ſum of the three ſides, and of its exceſs above BC, is to the rectangle contained by the ſines of BA, AC as the ſquare of the coſine of half BAC to that of the radius; and that the rectangle contained by the ſines of BA, AC, is to the rectangle contained by the ſines of half the ſum, and half the difference of the baſe and the exceſs of the ſides, as the ſquare of the radius to the
a 22. 5. ſquare of the ſine of half BAC; therefore, by equality [a], the rectangle contained by the ſines of half the ſum of the three ſides, and of its exceſs above the baſe, is to the rectangle contained by the ſines of half the ſum, and half the difference of the baſe, and the exceſs of the ſides, as the ſquare of the coſine of half the angle BAC to the ſquare of its ſine; that is, as the ſquare of the radius to the ſquare of the tangent of half the
b 2. Cor. Def. 10. P. T. angle BAC [b].

SOLUTION of the Cases of Oblique-angled Spherical Triangles.

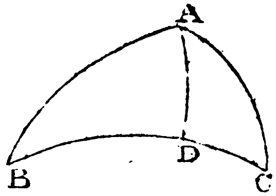
GENERAL PROPOSITION.

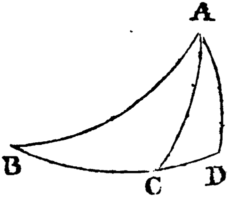
IN any ſpherical triangle, of the three ſides and three angles, any three being given, the other three may be found.

The

Sph. Trig.

The ſeveral caſes of this propoſition may be reſolved by the help of the three general proportions, together with the 22d, and any one of the ſubſequent propoſitions; as in the following





Table, in which ABC is any ſpherical triangle; and the perpendicular AD either falls within the triangle, or meets the baſe BC produced beyond C.

The caſes referred to, are thoſe of the preceding Table.

Caſes.	Given.	Sought.	Solution.
1	AB, AC and B oppoſite to AC.	C the angle oppoſite to AB.	Sin. AC : ſin. AB : : ſin. B : ſin. C. If the ſum of BA, AC be leſs than 180°, and AB leſs than AC; the angle at C is acute: Or, if the ſum of BA, AC be greater than 180°, and AB greater than AC; ACB is obtuſe. In other caſes, ACB is ambiguous.
2	AB, AC and B oppoſite to AC.	BC the third ſide.	R : cos. B : : tan. AB : tan. BD (caſe 2.) and cos. AB : cos. AC : : cos. BD : cos. DC. When ABC is acute, DC, CA are of the ſame affection, otherwiſe they are of contrary affection. If CD be not leſs than DB, their ſum is CB; if CD be leſs than DB, but their ſum not leſs than 180°, their difference is CB. In other caſes, CB is ambiguous.

Caſes.	Given.	Sought.	Solution.
3	AB, AC and B oppoſite to AC.	A the angle contained by the ſides.	R : cos. AB : : tan. B : cot. BAD, (caſe 3.), and tan. AC : tan. AB : : cos. BAD : cos. DAC. If B be acute, DAC and AC are of the ſame affection, otherwiſe they are of different affection. If DAC be not leſs than BAD, their ſum is BAC : if DAC be leſs than BAD, but their ſum not leſs than 180°, their difference is BAC. In other caſes BAC is ambiguous.
4	B, C and AB, two angles and the ſide oppoſite to one of them C.	AC the ſide oppoſite to B.	Sin. C : ſin. B : : ſin. AB : ſin. AC. If the ſum of B and C be leſs than 180°, and B leſs than C, AC is acute : or if the ſum of B and C be greater than 180°, and B greater than C, AC is obtuſe. In other caſes, AC is ambiguous.
5	B, C and AB, two angles and the ſide oppoſite to one of them C.	A the third angle.	R : cos. AB : : tan. B : cot. BAD, (caſe 3.), and cos. B : cos. C : : ſin. BAD : ſin. DAC, (p. 3.), which is leſs than BAD, if B, C be of diff. affection, or leſs than the ſupplement of BAD, if B and C be of the ſame affection : In other caſes it is ambiguous. When B and C are of the ſame affection, BAC is the ſum of BAD, DAC, otherwiſe it is their difference.

Caſes.

Sph. Trig.

Caſes.	Given.	Sought.	Solution.
6	B, C and AB, two angles and the ſide oppoſite to one of them C.	BC the ſide between the angles.	R : cos. B : : tan. AB : tan. BD, (caſe 2.), and tan. C : tan. B : : ſin. BD : ſin. DC; and DC is leſs than DB, if B and C be of diff. affection; or leſs than the ſupplement of DB, if B and C be of the ſame affection. In other caſes, DC is ambiguous. If B and C be of the ſame affection, BC is the ſum of BD, DC; otherwiſe it is their difference.
7	AB, BC, and B, two ſides and the included angle.	C one of the other angles.	R : cos. B : : tan. AB : tan. BD, (caſe 2.), and the difference of BC and BD is DC. And ſin. DC : ſin. DB : tan. B : tan. C, (p. 3.), and B, C are of the ſame affection, if BC be greater than BD; otherwiſe they are of diff. affection.
8	AB, BC, and B, two ſides and the included angle.	AC the third ſide.	Find BD and DC as in the laſt caſe, then cos. BD : cos. DC : : cos. BA : cos. AC (p. 3). If BD, DC be of the ſame affection, BA, AC are of the ſame affection; otherwiſe they are of different affection.
9	A, B, and AB, two angles and the included ſide.	C the third angle.	R : cos. AB : : tan. B : cot. BAD, (caſe 3.), and the diff. of BAC, BAD is DAC, then ſin. BAD : ſin. DAC : : cos. B : cos. C, (p. 3.), if BAC be greater than BAD, B, C are of the ſame affection; otherwiſe they are of different affection.
10	A, B, and AB, two angles and the included ſide.	AC one of the other ſides.	Find BAD and DAC, as in the laſt caſe; then cos. DAC : cos. BAD : : tan. AB : tan. AC. If DAC, B be of the ſame affection, AC is leſs than 90°; otherwiſe it is greater than 90°.

Caſes.	Given.	Sought.	Solution.
11	AB, AC, BC the three ſides.	B one of the angles.	Let the perp. AD fall within, or be the neareſt to B or C that falls without, tan. $\frac{1}{2}$ BC : tan. $\frac{1}{2}$ ſum of BA, AC : : tan. $\frac{1}{2}$ diff. of BA, AC : tan. $\frac{1}{2}$ E, and $\frac{1}{2}$ E added to $\frac{1}{2}$ BC, gives the ſegment neareſt the greater ſide, if the ſum of AB, AC be leſs than 180° ; otherwiſe it gives the ſegment neareſt the leſs ſide. (Prop. 22.). And tan. AB : tan. BD : : R : cos. B. (caſe 12.). Otherwiſe, Let D be $\frac{1}{2}$ the diff. of AB, BC ; then the rect. ſin. AB, ſin. BC : rect. ſin. ſum and diff. of D, and $\frac{1}{2}$ AC : : R^2 : : ſin^2. $\frac{1}{2}$B. (P. 23). Otherwiſe, Let P be $\frac{1}{2}$ the peremeter; then rect. ſin. AB, ſin. BC : rect. ſin. P. ſin. diff. of P, AC : : R^2 : cos^2, $\frac{1}{2}$ B. (24).
12	A, B and C the three angles.	AC one of the ſides.	With the ſupplement of either of the angles A, C, and the meaſures of the other two angles, ſuppoſe a triangle made ; and in it find the angle oppoſite to the ſide which is the meaſure of the angle at B, and the meaſure of the angle thus found is AC.

NOTES.

NOTES.

BOOK I.

DEFINITIONS.

Book I.

THE 1ſt definition wants a condition to make it complete; for to have no magnitude is not peculiar to a point: This condition is now inſerted from Dr Hooke, who ſays, that a point has poſition, and a relation to magnitude, but has itſelf no magnitude: It may alſo be ſaid to be an indiviſible mark in magnitude, as Tacquet has it: Or, it may be ſaid to be a ſign uſed for determining poſition and the extremities of lines, for the name σημεῖον appears to have been given to it from its uſe.

The 8th definition is left out; becauſe it does not belong to the Elements; nor can it be explained, ſo as to be underſtood by beginners, as is obſerved by Dr Simſon.

The 13th definition is alſo omitted, becauſe it is uſeleſs in a tranſlation, its only deſign being to explain a Greek word.

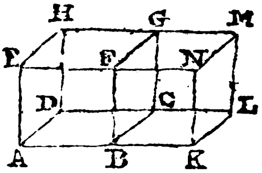
And the 19th, which is the definition of a ſegment, is left out here, becauſe it is given in the third book, which is its proper place.

And the definition of the radius of a circle is introduced, becauſe it is very frequently uſed by Geometers.

Theſe are all the alterations that have been made in the definitions of this book; but many more might have been made with propriety. The firſt nine definitions might have been given in the form of an introduction, for they are none of them geometrical, except the ſeventh, as amended by Dr Simſon. The terms by which a line and a ſuperficies are defined, give ſome explanation of the meaning of theſe words, but give no geometrical criteria by which to know them; and the beſt way of acquiring proper ideas of them, is by conſidering their relation to a ſolid, and to one another, as Dr Simſon has done.

"It is neceſſary to conſider a ſolid, that is, a magnitude which has length, breadth, and thickneſs, in order to underſtand aright the definitions of a point, line, and ſuperficies; for all theſe ariſe from a ſolid, and exiſt in it: The boundary, or boundaries which contain a ſolid, are called ſuperficies, or the boundary which is common to two ſolids, which are contiguous, or which divides one ſolid into two contiguous parts, is called a ſuperficies: Thus, if BCGF, be one of the boundaries which contain the ſolid ABCDEFGH, or which is the common boundary of this ſolid, and the ſolid BKLCFNMG, and is therefore in the one as well as the other ſolid, it is called a ſuperficies, and has no thickneſs: For if it has any, this thickneſs muſt either be a part of the thickneſs of the ſolid AG, or the ſolid BM, or a part of the thickneſs of each of them. It cannot be a part of the thickneſs of the ſolid BM; becauſe, if this ſolid be removed from the ſolid AG, the ſuperficies BCGF, the boundary of the ſolid AG remains ſtill the ſame as it was. Nor can it be a part of the thickneſs of the ſolid AG; becauſe, if this be removed from the ſolid BM, the ſuperficies BCGF, the boundary of the ſolid BM, does nevertheleſs remain; therefore the ſuperficies BCGF has no thickneſs, but only length and breadth.

The boundary of a ſuperficies is called a line; or a line is the common boundary of two ſuperficies that are contiguous; or it is that which divides one ſuperficies into two contiguous parts: Thus, if BC be one of the boundaries which contain the ſuperficies ABCD, or which is the common boundary of this ſuperficies, and of the ſuperficies KBCL, which is contiguous to it, this boundary BC is called a line, and has no breadth: For, if it has any, this muſt be part either of the breadth of the ſuperficies ABCD, or of the ſuperficies KBCL, or part of each of them. It is not part of the breadth of the ſuperficies KBCL; for, if this ſuperficies be removed from the ſuperficies ABCD, the line BC, which is the boundary of the ſuperficies ABCD, remains the ſame as it was: Nor can the breadth that BC is ſuppoſed to have be a part of the breadth of the ſuperficies ABCD; becauſe, if this be removed from the ſuperficies KBCL, the line BC, which is the boundary of the ſuperficies KBCL, does nevertheleſs remain; therefore the line BC has no breadth: and becauſe the line BC is in a ſuperficies, and that a ſuperficies has no thickneſs, as was ſhewn; therefore a line has neither breadth nor thickneſs, but only length.



The boundary of a line is called a point, or a point is the common boundary or extremity of two lines that are contiguous:

Thus, if B be the extremity of the line AB, or the common extremity of the two lines AB, BK, this extremity is called a point, and has no length: For, if it has any, this length muſt either be part of the length of the line AB, or of the line BK. It is not part of the length of the line KB; for, if the line KB be removed from AB, the point B, which is the extremity of the line AB, remains the ſame as it was: Nor is it part of the length of the line AB; for, if AB be removed from the line KB, the point B, which is the extremity of the line KB, does nevertheleſs remain; therefore the point B has no length: and becauſe a point is in a line, and a line has neither breadth nor thickneſs; therefore a point has no length, breadth, nor thickneſs. And in this manner, the definitions of a point, line, and ſuperficies, are to be underſtood."

No definition of a ſtraight line has been given that is unexceptionable, though many of the ancients attempted it, as Proclus obſerves, who has alſo preſerved their definitions. That given in the Elements, viz. its lying evenly, equally, or uniformly between its extremities, expreſſes the nature of a ſtraight line too metaphyſically: its meaning is, that a ſtraight line has not a convex and a concave ſide; but that both ſides are alike. But the diſtinguiſhing character of a ſtraight line, according to Euclid, is, that it is impoſſible to apply one part of it to another, or one ſtraight line to another, without their coinciding. All other lines require ſome artifice in applying them to one another, in order to make them coincide; but no ſuch artifice is neceſſary in the caſe of ſtraight lines, for they always coincide in whatever way we proceed to apply them to one another. That this was Euclid's idea of a ſtraight line, is manifeſt from the fourth and eighth propoſitions of the firſt book, in which he does not ſhew how the ſides of the triangles are to be applied to one another, ſo as to coincide, but takes it for granted, that AB in the fourth ſhall lie along DE, and that BC in the eighth ſhall lie along EF, as ſoon as they are applied to one another. Now, this facility of application follows immediately from the uniformity of the ſides of a ſtraight line, and the other properties of a ſtraight line are eaſily deduced from it.

Plato's definition, that the extremity of a ſtraight line caſts a ſhadow along the whole line; and that of Archimedes, that a ſtraight line is the leaſt of all the lines which have the ſame extremities with it, were evidently deſigned for particular purpoſes, and are not fit for the Elements.

The other definitions mentioned by Proclus, may be reduced to two. One of them is, that a ſtraight line is that of which the poſition is determined by the poſition of any two points of it. This is a property of ſtraight lines which is ſuppoſed in the firſt

Book I. and ſecond poſtulates, for we certainly underſtand by theſe poſtulates, that only one ſtraight line can be drawn between the ſame two points, and that a terminated ſtraight line can be produced only in one direction; ſo that the poſition of the points determines the poſition of the line; it is therefore probable, that Euclid underſtood it as included in his tenth axiom; for that it was not his definition, is evident from what has been ſaid, and becauſe it is his 26th propoſition of the Data, which is the 29th of Simſon's edition. Beſides, it gives too remote an intimation of the nature of ſtraight lines, and their properties cannot be eaſily deduced from it.

The other definition is, that a ſtraight line is that which cannot cut another ſtraight line in more points than one. In the Greek, this is given in the firſt propoſition of the eleventh book, as the reaſon why two ſtraight lines cannot have a common ſegment; and this laſt is aſſumed every where in the Elements. Their cutting only in one point ſeems to be a particular caſe, or at leaſt a conſequence of that facility of coinciding which, as was obſerved, Euclid makes the criterion of ſtraight lines; it may therefore very properly be made an axiom: and it ſeems neceſſary to do this, becauſe, when we are not actually applying ſtraight lines to one another, it may be found that two points of one of them coincide with two points of another, in which caſe the coinciding of the lines follows more immediately from the axiom, than from their uniformity: and it is probable, that Euclid underſtood his tenth axiom in all this extent: But to make it a definition, would be attended with diſadvantages, for then the only criterion of the coinciding of ſtraight lines, would be their meeting in two points: To ſuppoſe that they coincide, without firſt finding that they meet in two points, would be to aſſume a property of them, not contained in the definition: We are not therefore warranted to ſay, as in the fourth propoſition, Let the triangle ABC be applied to DEF, ſo that the point A may be on D, and the ſtraight line AB upon DE; for all that follows from AB and CD being ſtraight lines, according to this definition, is, that they would coincide, if, beſides A being on D, ſome other point of AB be ſhown to coincide with ſome other point of DE. Farther, the property contained in this definition is not the ſame with the principle upon which the mechanical deſcription is founded, as might eaſily be made manifeſt; not to mention its being negative, and not derived from inherent properties.

In like manner, no geometrical definition of an angle has been given; and accordingly, among the ancients, different definitions were given by different authors. Thus, Euclid called it κλισις, the inclination of lines: Apollonius called it συναγωγη, the drawing together of a ſuperficies to a point; and others called it the firſt diſtance of the containing lines: And the philoſophers diſputed whether it was a quantity, or a quality, or a relation, or whether it belonged to one or to ſeveral of the categories. The moderns have generally followed Euclid, or have called it the aperture or opening of two lines that meet. But no author has ever deduced the properties of angles from his definition; nor is it eaſy to do ſo without metaphyſical reaſoning. Thus we find, that nothing in the Elements depends upon theſe nine definitions; and therefore the purpoſe of inſtruction would be better obtained by removing them from the definitions, and explaining them in an introduction.

Beſides theſe, there are ſeveral other definitions which might have been omitted, or altered for the better. Thus, in the definition of an equilateral triangle, there is nothing peculiar to a triangle, for every rectilineal figure is in the Elements ſaid to be equilateral, when it has all its ſides equal. And the acute and ſcalene triangles are never mentioned in the Elements, nor have they any peculiar properties: They may indeed be ſaid to be neceſſary for completing the arrangement of triangles; but even this cannot be pleaded for the definitions of quadrilaterals, for the claſſification of them in the Elements is complete, and altogether different from that in the definitions; ſo that the definitions are not well adapted to the work itſelf, and might have been altered with advantage.

It ſhall only be farther remarked, that the whole deſign of definitions is, to give names to things that have certain properties; and that unleſs the properties aſcribed to them be their generation, the definitions give no foundation from which to conclude the exiſtence of the things defined, as Dr Barrow obſerves. It is not therefore regular to deduce corollaries immediately from definitions, becauſe corollaries, as well as other propoſitions, ſuppoſe the exiſtence of the things concerned in them. Accordingly, Euclid either aſſumes their exiſtence in his poſtulates, or exhibits it in a conſtruction, or deduces it from what has been previouſly eſtabliſhed, before he deduce any of their properties from their definitions.

AXIOMS.

THE only alteration that is made here, is in the 10th axiom, which is now made more extenſive than it was before.

It includes not only the axiom, That two ſtraight lines cannot incloſe a ſpace, but alſo the other, That two ſtraight lines cannot have a common ſegment: This laſt is every where ſuppoſed in the Elements, though it be no where repeated, except in the firſt propoſition of the eleventh book.

For example, it is aſſumed in the ſecond poſtulate; for, as Proclus obſerves, that poſtulate would be nugatory, if the terminated ſtraight line could be the common ſegment of two ſtraight lines. In like manner, as Zeno objects, the firſt propoſition is not ſufficiently demonſtrated, without aſſuming, that two ſtraight lines cannot have a common ſegment: For if the ſtraight lines AC, CB can have a common ſegment, the ſides of the triangle may be only parts of them, between its extremity and the points A and B; and therefore they may be leſs than AB; ſo that the triangle may not be equilateral. Likewiſe, without this ſuppoſition, the fourth would not be ſufficiently demonſtrated; for when AB is applied to DE, if they could have a common ſegment leſs than AB, they would coincide the length of this ſegment; but not afterwards; ſo that the point B might not coincide with the point E.

The fifth depends on the ſame ſuppoſition; for if either ſide of the iſoſceles triangle, as AB, be the common ſegment of two ſtraight lines, theſe lines would make unequal angles with the baſe BC on the other ſide of it: But each of them would be equal to the angle BCE, by the fifth prop. In the ſame manner, it may be ſhown, that the ſame thing is ſuppoſed in many other propoſitions; and therefore it is evident that Euclid conſidered it as an axiom.

As to axioms, it is generally allowed, that they ought to be ſelf-evident, and likewiſe as few as poſſible; for nothing ought to be conſidered as ſelf-evident that is capable of demonſtration.

The poſtulates ought likewiſe to be as few as poſſible; for, as Sir Iſaac Newton obſerves, poſtulates are principles which geometry borrows from the arts, and its excellence conſiſts in the paucity of them. The poſtulates of Euclid are all problems derived from the mechanics; but there are things aſſumed by geometers, which are neither ſelf-evident nor problems; ſuch as that called the 12th axiom: and it is plain, that ſuch aſſumptions ought not to be made, but when the thing aſſumed is demonſtrated elſewhere, or when it cannot poſſibly be demonſtrated; for all certainty ariſes from ſelf-evidence. Poſtulates likewiſe limit the geometer in his definitions, for that property of a line or figure on which its mechanical deſcription is founded, is the only property from which the aſſumer muſt define it; otherwiſe his conſtructions would not agree with his demonſtrations. Thus Euclid could not define his circle from any property but that of its having equal radii, becauſe it is deſcribed with the ſame opening of the compaſſes. And becauſe a ſtraight line is drawn either by ſtretching a thread, or by applying another ſtraight line to the place where it is to be drawn, Euclid's character of ſtraight lines muſt be the facility of applying them to one another, or the property of a ſtretched thread, which is its invariably aſſuming the ſame poſition, as often as it is ſtretched in the ſame ſituation.

Many of the moderns complain, that Euclid, by taking ſo little for granted, has narrowed the foundation of geometry too much. And they maintain, that we are at liberty to aſſume any thing, the exiſtence of which does not imply a contradiction, or involve an impoſſibility. But were this the caſe, we might aſſume every thing, for we certainly ſuppoſe nothing impoſſible, when we aſſume, that ſimilar rectilineal figures are to one another in the duplicate ratio of their homologous ſides; for Euclid has proved this to be certainly true. They perhaps mean things evidently poſſible; but many things may appear evident to a writer, which are not ſo to a learner. An author of Elements ought to conſider the ſtate of his reader's mind, who, being altogether unacquainted with geometry, can admit nothing to be ſelf-evident that is peculiar to it. It is neceſſity, and not liberty, which is the cauſe of aſſumptions; and the writer who aſſumes what he is not obliged to aſſume, is not writing for his reader's inſtruction, but is endeavouring to beg fame, by humouring his reader's indolence.

It follows from this, that aſſumptions ought to be as particular as poſſible; for example, we ought not to aſſume a general propoſition, if it can be demonſtrated by aſſuming a particular caſe of it, becauſe this would be to aſſume more than what is neceſſary.

Euclid never ſuppoſed any thing to be poſſible which he has not before ſhown to be poſſible; but this was not merely to avoid impoſſibilities, as ſome alledge, but to ſecure evidence, and to make his reader as certain of his concluſions as he himſelf was.

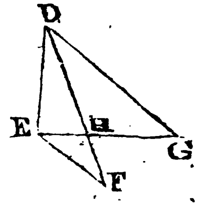
PROPOSITIONS.

In the 7th, the literal tranſlation of the enunciation is very obſcure; there is therefore a little alteration of the expreſſion introduced, but ſo as to preſerve the ſame meaning.

Proclus objects to the 12th, that there may be more than one perpendicular from C to AB, becauſe the circle may cut it in more than two points; the 17th limits the perpendicular to one, and therefore the proper place of this 12th is after the 17th. And it is not uſed before the 12th of the ſecond book.

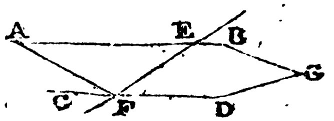
To the 15th there is added another corollary, which is as frequently uſed as the 14th, and is neceſſary in the conſtruction of the 14th and 15th of the ſixth book.

In the conſtruction of the 22d, it is now ſhewn that the circles cut one another, as Dr Simſon has done in his notes.



In the figure to the 24th, it may be ſhewn, as Dr Simſon has done in his notes, that the point F falls below the line EG; or thus: Let DF meet EG in H: and becauſe DHG is the exterior angle of the triangle DEH, it is greater [a] than the angle DEH: but the angle DEH is not leſs [b] than DGE, becauſe DG is not leſs than DE; therefore the angle DHG is greater than DGH; and DG, that is, DF is therefore greater [c] than DH; wherefore F is below EG.

a 16. 1.
b 5. or 18. 1.
c 19. 1.



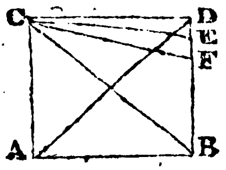
The 27th may be demonſtrated without the help of the 16th, thus: Let EF fall on AB and CD, and make the alternate angles AEF, EFD equal; AB is parallel to CD. If not, let them, if poſſible, meet towards B, D, in G; and make EA equal to FG, and join AF: and becauſe AE, EF are equal to GF, FE, and the angle AEF equal to GFE, the baſe AF is equal [a] to EG, and the angle AFE to the angle GEF; and the angle AEF is equal to EFG; therefore the two angles AFE, EFG are equal to AEF, FEG, that is, to two right angles [b]; wherefore AF is in the ſame ſtraight line with FG; and the two ſtraight lines AFG, AEG meet in more than one point, which is impoſſible [d]: Therefore AB and CD do not meet, though produced; wherefore they are parallel [e].

a 4. 1.
b 13. 1.
d 10. Ax. 1.
e 35. Def. 1.

The 29th depends on the 12th axiom, which is not ſelf-evident; and has given much to do, both to ancient and modern geometers. Some have attempted to ſubſtitute another axiom in its place; or to give another definition of parallel ſtraight lines; but their attempts have not been ſucceſsful. Others have attempted to demonſtrate the 12th axiom; but when they have done this, by aſſuming a new axiom, the evidence of their axiom has been objected to; and when they have attempted it without a new axiom, they have hitherto failed. It was therefore thought neceſſary to demonſtrate it without a new axiom; and ſince this is the firſt time that it has been accompliſhed, it is hoped the attempt will be acceptable to accurate geometers, who know the importance of having every thing in geometry eſtabliſhed on a ſolid baſes.

PROP. I.

If two equal ſtraight lines AC, BD on the ſame ſide of the ſtraight line AB, make the interior angles CAB, ABD equal to one another; the ſtraight line CD which joins their extremities, makes with them angles ACD, CDB equal to one another.



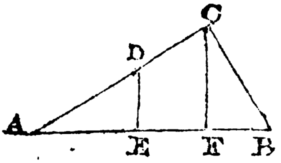
Join AD, BC: and becauſe in the triangles CAB, ABD, CA, AB, are equal to DB, BA, and the angle CAB equal to ABD; the baſe CB is equal [a] to AD: and in the triangles ACD, BDC, AC, CD are equal to BD, DC, and the baſe AD is equal to BC; therefore the angle ACD is equal [b] to the angle BDC. a 4. 1. b 8. 1.

Cor. Hence, if AC, BE make equal angles CAB, ABE with AB, but the angle BEC be greater than ACE; the ſtraight line AC is greater than BE.

For, if AC were equal to BE, the angle BEC would be equal to ACE, but it is not. If AC be leſs than BE, make BF equal to AC, and join CF; therefore the angle BFC is equal to ACF; but BFC is greater [c] than BEC; therefore BCF, and much more BCE, is greater than BEC; and it is alſo leſs; which is impoſſible: Wherefore BE muſt be leſs than AC. c 16. 1.

PROP. II.

If the quadrilateral figure BCDE, which has the angles at C, E right angles, have the ſides BE, CD produced, and meeting in A; the ſide BC is greater than the ſide DE.



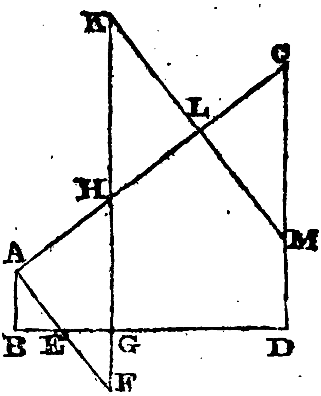
From C draw [a] CF perpendicular to AB: AF is greater than AE. For if CF fall on E, the angle CEB would be equal to DEB, each being a right angle, the leſs to the greater; which is impoſſible: and if AF be leſs than AE, and CE be joined, CFE would be a triangle, of which the angles CFE, CEF are together greater than two right angles; which is impoſſible [b]; therefore AF is greater than AE: and becauſe CFB, CBF are leſs [b] than two right angles, and CFB is a right angle, CBF is leſs than a right angle; therefore CB is greater [c] than CF. In the ſame manner, it may be proved, that each of the angles ACF, ADE is leſs than a right angle: and the angles a 12. 3. b 17. 1. c 19. 1.

Book I. ADE, EDC are equal [d] to two right angles; therefore EDC is
greater than a right angle: and becauſe the angle DEF is equal
d 13. 1. to EFC, but EDC is greater than DCF; therefore CF is great-
e Cor. to 1. of them. er [e] than DE: but CB is greater than CF; much more, therefore, is CB, greater than DE.

PROP. III.

If from A, C two points of the ſtraight line AC, perpendiculars AB, CD be drawn to BD; and the angle BAC be greater than a right angle; DCA is leſs than a right angle.

a 11. 1. Draw [a] AE at right angles to AC, and let it meet BD in E;
and make EF equal to EA, and EG to EB, and join FG; and
becauſe AE, EB are equal to
FE, EG, and the angle AEB
b 15. 1. is equal [b] to FEG; therefore
c 4. 1. the angle EGF is equal [c] to
ABE, that is, to a right angle. In the ſame manner, if FG meet AC in H, and HK be made equal to HF, and HL to HA, and KL be joined, it may be proved, that KL is at right angles to AC: and becauſe AEGH is a quadrilateral, of which HAE, HGE are right angles, and AE, HG meet in F; therefore AH is
d 2. of this. greater [d] than EG or EB. In
the ſame manner, it may be ſhown, that every perpendicular cuts off from AC and BD, a ſegment not leſs than any of the former; therefore, if this be done continually, ſome of the perpendiculars to AC ſhall meet CD: Let KL meet CD in M: and becauſe CLM is a right
e 17. 1. angle, LCM or ACD is leſs [e] than a right angle.



PROP. IV.

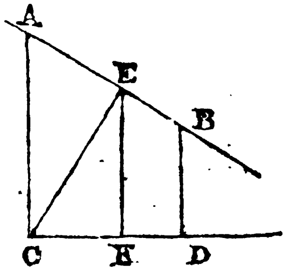
If the ſtraight line AC meet the two ſtraight lines AB, CD, and make the angle ACD a right angle, and CAB an acute angle; any number of perpendiculars to CD can be drawn from points in AB, all of them making acute angles with AB towards B.

a 12. 1. From C draw [a] CE perpendicular to AB: it ſhall fall to-
wards B, for if it ſhould fall on the other ſide of A, the angle
c 13. 1. CAE would be obtuſe [c], and therefore the angles CAE, AEC
would

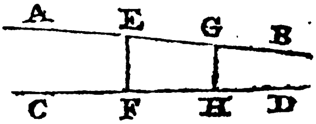
Book I.

would be greater than two right angles, which is impoſſible [b]; therefore CE falls towards B: From E draw [a] EF, perpendicular to CD; and it may be ſhown, as before, that it falls towards D: and becauſe CEB is a right angle, FEB is leſs than a right angle. In the ſame manner, may any number of perpendiculars to CD be drawn on the ſide of EF towards B; all of them making acute angles with AB towards B.

b 17. 1.



Cor. Hence, if EF be perpendicular to CD, from the point E in AB; and if all the perpendiculars to CD that can be drawn from points in AE, make acute angles with AB towards B, the angle FEB is an acute angle. For any number of perpendiculars to CD, can be drawn from points in AB nearer to B, than any that are on the other ſide of E, all making acute angles towards B; ſome of theſe points ſhall therefore be nearer to B, than the point E: Let GH be perpendicular to CD, from G nearer to B than the point E, and making HGB an acute angle; therefore HGE is greater [c] than a right angle: and becauſe EF, GH are perpendicular to CD, and HGE is greater than a right angle; therefore FEG is leſs [d] than a right angle.



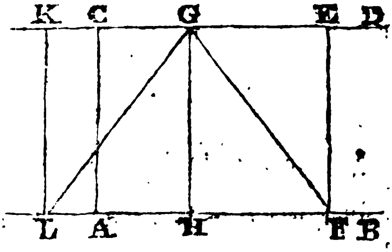
c 13. 1.

d 3. of this.

PROP. V.

If two ſtraight lines AB, CD be at right angles to the ſame ſtraight line EF; any ſtraight line CA at right angles to one of theſe lines AB, is alſo at right angles to the other CD.

For the perpendiculars to AB that can be drawn from points in CE, cannot all make acute angles with CD towards D, becauſe then FED would be an acute angle [a]; but it is not: let therefore GH be perpendicular to AB, and HGE not be an acute angle: But neither is it an obtuſe angle, becauſe then GEF would be an acute angle [b]; but it is not; therefore HG muſt be at right angles to CD. Make GK equal to GE, and HL to HF, and join KL, LG, GF: and



a Cor. 4. of this.

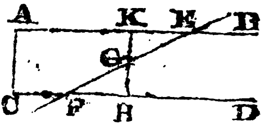
b 3. of this.

and becauſe in the triangles GHL, GHF, LH, HG are equal to FH, HG, and the angles at H right angles; therefore the baſe
c 4. 1. GL is equal [c] to GF, and the angle HGL equal to HGF, and GLH to GFH: but the whole angle HGK is equal to HGE, therefore the remaining angles LGK, FGE are equal: and becauſe in the triangles GKL, GEF, KG, GL are equal to EG, GF, and the angle KGL to EGF; the baſe LK is equal to EF, and the angles GKL, GLK to the angles GEF, GFE [c]; but GEF is a right angle, therefore GKL is a right angle; and becauſe the angle GLH is equal to GFH, and GLK to GFE, the whole HLK is equal to the whole HFE, that is, to a right angle. In the ſame manner, if ſegments of AB, CD be taken equal to FL, EK, it may be proved, that the ſtraight line joining their extremities is at right angles to AB, CD; and ſo on: Therefore a ſtraight line can be found on the other ſide of AC, that is, at right angles to AB, CD: let this be KL: and if ACE, ACK
d 13. 1. be not right angles, one of them is greater [d] than a right angle:
b 3. of this. let this be ACK; therefore LKC is leſs [b] than a right angle: and it is alſo a right angle; which is impoſſible: Therefore CA is at right angles to CD.

PROP. VI.

If a ſtraight line EF meet two ſtraight lines AB, CD which are at right angles to ſome ſtraight line AC; the alternate angles AEF, EFD ſhall be equal to one another.

If EF be at right angles to one of them, it is alſo at right
a 5. of this. angles [a] to the other. But if not, biſect [b] EF in G, and draw [c]
b 10. 1. GH perpendicular to CD; it is alſo per-
c 12. 1. pendicular to AB [a]: let it meet AB in K: and becauſe the angles EGK, FGH
d 15. 1. are vertical, they are equal [d]; and the angles at H, K are right angles; there are therefore two angles of the triangle EGK, equal to two of the triangle FGH; and the ſides EG, GF oppoſite to equal angles are equal; therefore the third angle KEG is equal to the third angle HFG.

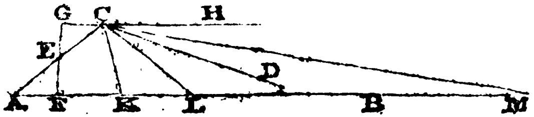


PROP. VII.

If two ſtraight lines AB, CD be cut by a third AC, ſo as to make the interior angles BAC, ACD on the ſame ſide of it, together leſs than two right angles; AB and CD being produced, ſhall meet one another towards the parts on which are the two angles, which are leſs than two right angles.

Let CAB be the leſſer of the two angles BAC, ACD, if they be unequal; and biſect AC in E [a], and draw [b] EF perpendicular to AB, and from C draw [b] CG perpendicular to EF, and produce it to H: and becauſe AB, GH are at right angles to FG, and AC meets them; the angle BAC is equal [c] to ACG: to each of them add the angle ACH; then BAC, ACH are equal to ACG, ACH, that is, to two right angles [d]: But BAC, ACD are leſs than two right angles; therefore the angle ACH is greater than ACD; and CD falls between CH and AB. Make AK equal to AC, and join CK: and becauſe CK meets the two ſtraight lines AB, CH, which are at right angles to FG; the angle AKC is equal [c] to KCH: but the angle ACK is equal [e] to AKC, becauſe AK is equal to AC; therefore the angle ACK is equal to KCH. In like manner, if KL be made equal to KC, and CL joined, it may be proved, that it biſects the angle KCH; and ſo on. And becauſe the angles ACK, KCL are each of them greater than the angle DCH; and that the angle DCH taken ſome number of times, is greater than ACD; much more ſhall the ſame number of the angles ACK, KCL, &c. make an angle greater than ACD: let the angle thus made, which is greater than ACD, be ACM: and becauſe ACM is greater than ACD; CD falls within the angle ACM; and therefore, if it be produced, it muſt cut AB betwixt the points A and M.

a 10. 1.
b 12. 1.
c 6. of this.
d 13. 1.
e 5. 1.



Thus the 12th axiom is accurately demonſtrated: But the tediousneſs of the proceſs, and the alledged obſcurity of the propoſition, may perhaps be thought ſufficient reaſons for aſſuming a more obvious property of ſtraight lines: But when propoſitions are not ſelf-evident, authors are not agreed about what makes one of them more obvious than another. We come to the knowledge of many properties of figures, by experience and obſervation, before we be able to deduce them from ſelf-evident principles: and this is the reaſon of our more readily aſſenting to propoſitions in which lines alone are concerned, than to thoſe in which angles are concerned, for we are conſtantly uſing the former, but make few obſervations on the latter. But one perſon may be better acquainted with one property, and another with another property; which makes it difficult to determine which of them is the moſt obvious: and this is the reaſon why the

Book I. the moderns are divided in their opinions about it. The ancients did not admit experience to be judge of obviouſneſs in geometry: But when they were obliged to aſſume any thing, they firſt made it as probable as they could, by demonſtrating its converſe, and other ſimilar properties; after which they aſſumed it at once: and this method is better adapted to geometry than the other; for thus the author's ſcheme is open to the view of his reader, who can then judge for himſelf of the propriety of the aſſumption. But if an author aſſume any thing from his own experience, which is not made probable from any thing he has ſaid before; he obliges his reader to take it entirely upon his word.

From this it follows, that the proper place of the 12th axiom is between the 28th and 29th propoſitions, for it is not made ſufficiently probable before the 28th. And much of its obſcurity would be removed by placing it there; for neither it, nor any that the moderns have ſubſtituted inſtead of it, have the ſmalleſt title to be ranked with the axioms.

The axioms which the moderns have preferred to it, may be reduced to three, though they may be expreſſed in more ways than one.

1. That a ſtraight line perpendicular to one of two parallels, is alſo perpendicular to the other. Or it may be expreſſed thus: If a perpendicular to a ſtraight line drawn from a point of another, be not perpendicular to that other, the two lines will meet. Theſe two are evidently of the ſame order; for if parallels have a common perpendicular, ſtraight lines which have not a common perpendicular, are not parallel, that is, they meet; and if ſtraight lines which have not a common perpendicular do meet, it is evident, that thoſe which do not meet, or are parallel, muſt have a common perpendicular. This axiom was uſed by ſeveral authors of the laſt century: It may be demonſtrated by the fifth of the preceding propoſitions; and the 29th may be demonſtrated from it nearly as in the ſixth of this.

2. Perpendiculars to one of two parallels from points of the other, are equal to one another. Or, if two perpendiculars to a ſtraight line from points of another be unequal, theſe lines will meet. Theſe may be ſhewn to be of the ſame order as the former were: They may be proved by the 7th of the preceding; and the 29th may be proved from them, by firſt proving the former axiom, and then the 29th as in the ſixth of the preceding.

3. A ſtraight line which meets one of two parallels, meets alſo the other. Or, two ſtraight lines cannot be drawn through the ſame point, parallel to the ſame ſtraight line. One of theſe can be eaſily deduced from the other, as in the firſt: They may

be proved by the help of the 7th preceding; and then the 29th may be proved from them: or rather may be omitted altogether; for by confining the parallels through the ſame points to two, they determine the properties demonſtrated in the 27th and 28th, to be the only eſſential properties of parallels.

We might have mentioned Dr Simſon's axiom in the beginning of his demonſtration of the 12th axiom, which is certainly much more evident than any of the former; but is not ſo eaſily applied to the demonſtration of parallels.

Of all theſe axioms, the firſt is preferable, becauſe it can be demonſtrated without the others, and the 29th may be proved from it with the greateſt eaſe. The ſecond can only be uſed to demonſtrate the firſt; and the third cannot be proved, without firſt proving the firſt or ſecond, or elſe the 12th axiom. The firſt is indeed a particular caſe of the 29th, or of the 12th axiom; but, as was obſerved, it is better to aſſume a particular caſe, than the general propoſition, and its truth is much more probable: It is likewiſe inferior to the ſecond, if experience be the proper judge of obviouſneſs; but this is counterbalanced by the eaſe with which parallels are demonſtrated by it; for they would thus be made as ſimple as it is poſſible to make them.

Of the 35th a more general demonſtration is now given, by uſing a method of ſubtraction, which was introduced into this propoſition, by Dr Simſon, from M. Clairault's Geometry; and is one of the moſt general and uſeful applications of the third axiom.

The conſtructions of the 37th and 38th are a little altered, to avoid proving that the ſtraight lines drawn parallel to the ſides of the triangles muſt meet the line joining their vertices, which was neceſſary, according to the former conſtructions.

In the 39th, it is now proved, by the 12th axiom, that the ſtraight line drawn through the vertex of one of the triangles, parallel to the baſe, muſt meet a ſide of the other triangle, as was neceſſary here, becauſe this is the firſt propoſition in which it is required that the lines ſhould meet. In the Greek, the demonſtration is omitted the firſt ſix times, viz. in the 37th, 38th, 39th, 40th, 42d, and 44th; but is given fully in the ſeventh time, in the ſame 44th, after which it is omitted throughout the Elements, except in the 10th of the ſecond book and the 4th of the ſixth, where it is as obvious as in any of the numerous places in which it is omitted. Dr Simſon takes notice of the omiſſion in one place, and concludes from it, that the demonſtration has been ſpoiled, and the ſame may be ſaid of the reſt: To remove this irregularity, without embarraſſing the demonſtrations, a corollary is added to

Book I. this propoſition, on which the meeting of a line with parallels depends more immediately than on the 12th axiom.

In the conſtruction of the 45th, it was required, that the parallelogram GM have an angle equal to E, from which it is proved, that KH is in the ſame ſtraight line with HM: But this is eaſier done by producing KH to M; and then it is only neceſſary that the parallelogram GM have an angle equal to GHM, or have GHM for one of its angles.

BOOK II.

Book II. THERE are corollaries added to the 4th, the 5th, the 7th, the 8th, and the 10th propoſitions, which are chiefly intended to expreſs the enunciations of theſe propoſitions in a more general way, as is done by the moderns.

The demonſtration of the 8th is made ſhorter, by proving BK to be equal to KR, and BN together with PL to be equal to each of the rectangles AK, MR, KF.

The conſtruction of the 10th is now made ſimilar to that of the 9th, by which means they may both be demonſtrated in the ſame words, and therefore it was needleſs to repeat the demonſtration. In the Greek, it is firſt aſſumed, that DF meets a parallel to AD; and then it is demonſtrated, that EB meets a parallel to EC. They are caſes that have often occurred, and therefore their demonſtration might have been omitted, ſince it ought to have been given before; but if either of them be demonſtrated, it ought undoubtedly to have been the firſt mentioned.

Dr Simſon has divided the 13th into three caſes; but the two firſt can be demonſtrated in the ſame words: and the demonſtration of the third is unneceſſary, becauſe it is manifeſt from the 47th of the firſt book.

There are three propoſitions added to this book, becauſe they are very often uſed by geometers, and are connected with this part of the ſubject.

BOOK

BOOK III.

DEFINITIONS.

THE firſt definition is left out, becauſe it is not a definition, but a theorem, as Dr Simſon has ſhown.

The 7th, which is the angle of a ſegment, is left out, becauſe it can be of no uſe, and has done much miſchief.

And the definitions of an arch, and of a chord, are inſerted, becauſe they are continually uſed by geometers.

PROPOSITIONS.

In the 2d, the ſuppoſition by which the demonſtration was made indirect, is now left out; and it is then direct and complete; for it follows immediately from the definition of a circle, that a point between the centre and the circumference is within the circle; and this is directly proved in the demonſtration, when it is ſhown, that DE is leſs than DF, or that E is between D and F. And it is certainly as evident, that the point E in the radius DF is within the circle, as it is that a point without the circle is in the radius produced, which is aſſumed in the indirect demonſtration.

The 5th and 6th have the ſame conſtruction and demonſtration. Beſides, it is aſſumed in the 6th, that two circles can touch one another, which is not evident, until the 7th has been demonſtrated. The poſſible exiſtence of things is not to be aſſumed, any more than their properties. Euclid aſſumes the exiſtence of a plane, a ſtraight line, and a circle; but before he ſuppoſes any thing elſe to be poſſible, he either exhibits it in a conſtruction, or makes it evident by an example, which is generally given in the moſt ſimple caſe. Thus, he gives an example of a triangle, and conſequently of an angle in the firſt propoſition, and of a perpendicular in the 11th, and of parallels in the 27th, and of a parallelogram in the 33d, and of a ſquare, and conſequently of a rectangle, in the 46th, and this always before he aſſume them in an hypotheſis. This is the general rule which regulates the arrangement of the whole Elements, and it is in a great meaſure to it that they are indebted for their ſuperior beauty and elegance: and wherever it is departed from, it is owing either to the inſertion of a propoſition which is not Euclid's, or to a change in his arrangement: Which ever of theſe be the caſe with this propoſition, and the fourth of the fifth book, which are the only places in which the rule is tranſgreſſed, it is proper to remove the blemiſh: and this is now done here by putting the word "meet," inſtead of "cut," into the enunciation of the 5th, and by throwing out the 6th, becauſe it is then included in the 5th.

There ſeems to have been a ſeries of alterations introduced into the Elements, from the 9th to the 13th, by ſome editor, who intended either to change Euclid's arrangement, or to exclude the 7th and 8th from the Elements, on account of their length; and who did not perceive that the hypotheſis of the 11th, 12th, and 13th, depended on the 7th and 8th, as well as their demonſtrations. The firſt demonſtrations of the 9th and 10th, are generally allowed not to be Euclid's, for they depend upon a principle which he no where makes uſe of; and he could have demonſtrated them eaſier, even without the 7th and 8th, by the help of the 24th of the firſt book: and if the author of theſe two demonſtrations gave alſo a demonſtration of the 11th, we may conclude, that he would have placed it before that of Euclid: But the ſecond demonſtration of the 11th is very much vitiated, for part of it is omitted, and part of the firſt demonſtration is tranſcribed into it. This, however, is probably owing to the careleſſneſs of tranſcribers, who have made the 9th to depend upon the firſt part of the 7th, inſtead of the laſt part of it, and left out the word "circumference" in the 10th: and the ſhortneſs of the firſt demonſtration of the 11th, might have made the corruptions of the ſecond to be leſs attended to. This ſecond demonſtration has two caſes; one when the centre of the greater circle is within the leſs, and the other when it is without it: and this diſtinction of caſes can be of no uſe, unleſs the demonſtration depend upon the 7th or 8th: conſequently, we are led by this clauſe to the following demonſtration. Firſt, Let F be within the circle ADE: and becauſe F is a point in the diameter of the circle ADE, which is not the centre, FD which paſſes through the centre is greater than FA: but FH is equal to FA; therefore FD is greater than FH; which is abſurd. In like manner, it may be proved, by Prop. 8. that the ſame abſurdity would follow, if the point F be without the circle ADE. Therefore, &c. This appears to be Euclid's demonſtration; and if we uſe the point G, inſtead of F, to avoid the diſtinction of caſes, it is the moſt natural, eaſy, and elegant demonſtration, that can well be conceived. And the 12th can be demonſtrated in the ſame way. All theſe corrections are now made: and the ſecond caſe of the 13th is rejected, becauſe Dr Simſon's demonſtration of the firſt caſe is general, whether the circles touch on the inſide or the outſide.

A direct demonſtration is given of the 16th, for the reaſons given in the notes on the ſecond. It is evident, that the ſecond part is not Euclid's, but is put in by ſome editor, probably the ſame that introduced what we find in this and the 31ſt, about the angle of a ſegment; becauſe its only uſe is to ſhew, that but one ſtraight line can touch the circle in the ſame point: and this is alſo the only uſe of the 18th and 19th; for perpendiculars being fully determined in the firſt book, it cannot be ſuppoſed, if Euclid had proved in the 16th, that the perpendicular to the radius at its extremity is the only line that touches the circle there, that he would alſo have proved, in the 18th, that this radius is perpendicular to the tangent.

A more general demonſtration is given of the 21ſt, ſimilar to what Dr Simſon gives of the ſecond caſe, for that given in the Greek is applicable only to angles in a ſegment greater than a ſemicircle.

A very uſeful corollary is added to the 22d, and another to the 35th.

There are four propoſitions added to this book, of which the two firſt and their corollaries contain the converſes of the moſt uſeful propoſitions in this book, which are as often uſed as the propoſitions themſelves: and the other two contain very uſeful properties, the firſt of the circle, and the other of the triangle.

BOOK IV.

Book IV.

IT is now demonſtrated by the 12th axiom, that the ſides meet one another, ſo as to form the triangle LMN about the circle in the 3d propoſition; and that the perpendiculars DF, EF in the 5th meet one another; and that the ſtraight lines touching the circle in the 12th meet one another; becauſe theſe caſes are not ſimilar to that for which the corollary to the 39th of the firſt book was given.

It is likewiſe demonſtrated in the third, that the point L in which LM, LN meet falls above A, C; for if they were to meet below MN, the triangle thus conſtructed would not be deſcribed about the circle ABC, according to the fourth definition: Nor would it be equiangular to DEF.

BOOK V.

DEFINITIONS.

THE 3d and 8th are rejected, becauſe they are unneceſſary and uſeleſs; and the 9th is included in definition D.

And there are ſix new definitions introduced; the firſt of which relates to the preſent method of expreſſing the 5th and and 7th definitions: but the reſt ought to have been in the Elements before; and the want of them is one of the cauſes of the obſcurity in which the ſubject of proportion has been hitherto involved.

In order to underſtand the 5th and 7th definitions, it is neceſſary to conſider in what way we acquire our ideas of ratios.

Some geometers ſeem to think, that our ideas of the proportion of magnitudes are derived from our ideas of the nature of abſtract numbers. But we certainly acquire our firſt ideas of proportion from external objects, in the ſame manner that we acquire our ideas of numbers. By obſerving one magnitude to be double of another, we acquire the idea of a particular ratio, or relation, that the greater has to the leſs: and when we afterwards find two magnitudes, one of which is alſo double of the other, we ſay, that they have the ſame ratio which the two former have to one another, or that the four are proportionals. In like manner, by obſerving one magnitude to be triple, quadruple, or any multiple of another, we acquire ideas of other ratios: and by proceeding in this way, we obtain ideas of all the ratios belonging to the firſt claſs of commenſurable magnitudes; that is, when the greater is a multiple of the leſs. So that all theſe ratios are obtained by conceiving the leſſer magnitudes to be added to themſelves ſome number of times. Nor is it more difficult to conceive how we may obtain ideas of all the ratios belonging to the ſecond claſs of commenſurable magnitudes; that is, thoſe of which the greater is not itſelf a multiple of the leſs, but of which ſome multiple of the greater is alſo a multiple of the leſs. It is only by conceiving the greater to be added to itſelf continually, until the multiple contain the leſs exactly; and then the ratio of the greater to the leſs is obtained. For the ratio of A to B is determined, by ſaying, that four times A is equal to ſeven times B, as properly as by ſaying, that A is greater than B, by three fourths of B. In this manner, we may acquire ideas of the ratios of all magnitudes which have common multiples. The method may appear to be tedious, but it is ſimple and obvious.

Hence it appears, that a ratio is determined when a certain multiple of the greater is found to be a multiple of the leſs; and that this ratio is diſtinguiſhed from every other ratio, by the magnitudes to which it belongs, having theſe multiples equal; and therefore we conclude, that the ratio of two magnitudes is the ſame with the ratio of two other magnitudes, when ſome equimultiples of their antecedents are alſo equimultiples of their conſequents: But that if one of the equimultiples of the antecedents be a multiple of its conſequent, and the other not the ſame multiple of its conſequent; in that caſe, the ratios are not the ſame.

Again, whenever we find that there are ſome equimultiples of the antecedents, ſuch that one of them is leſs than a multiple of its conſequent, but the other not leſs than the ſame multiple of its conſequent; we need not inquire for equal multiples of the antecedents and conſequents, but may conclude, that though we ſhould find equimultiples of the antecedents, ſuch that one of them is a multiple of its conſequent, the other would not be the ſame multiple of its conſequent.

Let twice A be leſs than three times B, but twice C not leſs than three times D; then, although we ſhould find that three times A is equal to four times B, three times C is not equal to four times D. For twice C not being leſs than three times D; by tripling them, ſix times C is not leſs than nine times D; it is therefore greater than eight times D; and taking their halves, three times C is greater than four times D. In the ſame manner, if four times C were found to be equal to ſeven times D, it may be proved, that four times A is leſs than ſeven times B.

On the contrary, if three times A be equal to four times B, but three times C not equal to four times D; equimultiples of A and C can be found, ſuch that one of them is leſs, and the other not leſs, than equimultiples of B and D. If three times C be leſs than four times D, this is evident, for three times A is not leſs than four times B. But let three times C be greater than four times D by the magnitude E; then ſome number of times E ſhall exceed C; let this be three times E; and becauſe three times C is equal to four times D, together with E; by tripling them, nine times C is equal to twelve times D, together with three times E: and three times E is greater than C; therefore eight times C is greater than twelve times D: and three times A being equal to four times B; by tripling them, nine times A is equal to twelve times B; therefore eight times A is leſs than twelve times B; and eight times C is not leſs than twelve times D: Wherefore eight times A and eight times C are equimultiples of A and C, ſuch that one of them is leſs, and the other not leſs, than twelve times B and twelve times D, which are equimultiples of B and D.

Hence it is manifeſt, that if there be not ſome equimultiples of the antecedents, ſuch that one of them is leſs than a multiple of its conſequent; and the other not leſs than the ſame multiple of its conſequent, the magnitudes are ſo related to one another, that if one of the equimultiples of the antecedents be a multiple of its conſequent, the other is the ſame multiple of its conſequent: and therefore the magnitudes have the ſame ratio.

There is therefore only one caſe in which the ratios are not the ſame, viz. when one of the equimultiples of the antecedents is leſs than a multiple of its conſequent, and the other not leſs than the ſame multiple of its conſequent: and in this caſe, we ſay, that one of them is greater than the other; that is, we ſay, that the antecedent of which the multiple is not leſs than that of its conſequent, has a greater ratio to its conſequent than the other antecedent has to its conſequent: Or, becauſe the multiple of the former antecedent contains its conſequent oftener than the ſame multiple of the other antecedent contains its conſequent; we ſay, that the firſt ratio is greater than the other, when ſome multiple of the firſt antecedent contains its conſequent oftener than the ſame multiple of the other antecedent contains its conſequent.

And becauſe this is the only caſe in which the ratios are not the ſame, we conclude, that two ratios are the ſame, when taking any equimultiples whatſoever of the antecedents, they are either both leſs, or elſe both not leſs, than any equimultiples of their conſequents: Or, becauſe in this caſe the equimultiples of the antecedents contain their conſequents the ſame number of times, we ſay, that two ratios are the ſame, when all the equimultiples of the antecedents contain their conſequents equally.

This is a more ſimple expreſſion of the definition of equal ratios, than that given by Dr Simſon; for the literal tranſlation from the Greek, though ſhorter than his, is very obſcure: But the meaning is very nearly the ſame. In the Greek, the equimultiples of the antecedents are required to be both leſs, or both equal, or both greater, than the equimultiples of the conſequents; but here they are required to be both leſs, or both not leſs. It was, however, ſhewn before, that unleſs one of them be leſs, and the other not leſs, it will always happen, that if one of them be a multiple of its conſequent, the other is the ſame multiple of its conſequent; and therefore the definitions are the ſame: the generality of the multiples making all particular differences to vaniſh. For it is to be obſerved, that the multiples taken, either according to the definition now given, or according to that in the Greek, muſt be general, repreſenting any multiples whatſoever.

But, beſides ſimplicity of expreſſion, the definition now given has the advantage of the other, in its being, in moſt caſes, more eaſily applied to the purpoſe of demonſtration, as will be manifeſt to any one who chuſes to compare the demonſtrations of the 17th and 18th, or even of thoſe from the 7th to the 13th in this edition, with thoſe in the other editions. The demonſtrations depend on the ſame principles, and are conducted in the ſame manner, with thoſe in the former editions; but the ſimplicity of the conſtructions often renders fewer ſteps neceſſary in the demonſtrations.

PROPOSITIONS.

The 1ſt, 2d, and 6th, are made more general than they are in Euclid. They belong to all magnitudes which contain others equally, as well as to equimultiples of them.

That beginners have the 4th for the firſt propoſition that they read about proportionals, is one of the cauſes why they find this book ſo difficult to be underſtood by them. The words *multiple* and *equimultiples* are new to them, and it is with ſome difficulty that they can bring themſelves to believe, that theſe ſtrange words are expreſſions of ideas with which they are familiarly acquainted. But they begin to get over this difficulty, and to be reconciled to theſe words, when they ſee the uſe that is made of them in the axioms, and the three firſt propoſitions. In this propoſition, however, a new difficulty ariſes from the uſe that is made in it of the fifth definition. In order to expreſs that definition intelligibly in the Engliſh language, a good deal of circumlocution is neceſſary; and this renders it difficult for the reader, to connect the ſeveral parts of it, and to form a ſingle, complete, and juſt idea of the whole: and this difficulty is increaſed, rather than removed, by the abrupt way in which it is introduced into this demonſtration. Whatever idea the reader may have formed of the 5th definition itſelf, he muſt be allowed to be altogether unacquainted with the method of applying it to the purpoſe of demonſtration, for he has not yet had an opportunity of ſeeing it uſed that way. He muſt alſo be allowed to doubt the exiſtence of ſuch magnitudes as have all the properties mentioned in that definition, for this ought not to be taken for a poſtulate; and there are abundance of examples by which their poſſibility may be ſhewn, one of which is given in the 7th, and another in Prop. C. But they are not given before this propoſition; and until they be given, the mind muſt be allowed to doubt. Beſides, the reader's mind is preoccupied with the vulgar notion of proportion, which, though obſcure, is familiar, and attracts the attention whenever proportionals are mentioned. Now, the writer who would demonſtrate this propoſition as it ſtands, muſt overcome all theſe difficulties. He muſt ſhew, that the magnitudes aſſumed in the hypotheſis, are poſſible, and muſt place the dependence of every ſtep of the demonſtration upon the 5th definition, in the cleareſt point of view, ſo as to call the reader's attention from the vulgar to the accurate notion of proportionals, contained in that definition, and to compel his mind, upon admitting that definition, to admit the ſeveral concluſions drawn from it. Inſtead of this, Euclid aſſumes the poſſible exiſtence of four proportionals, a thing inconſiſtent with accuracy, and quite contrary to his method in all ſimilar caſes; and he draws his concluſions as implicitly as if his readers had been perfectly acquainted with the ſubject: It is therefore evident, that this is not the firſt time that he has uſed the fifth definition, and that the place in which this propoſition ſtands at preſent, is not the place which he had allotted to it. Beſides, it is placed at a diſtance from all the propoſitions about ratios, in the midſt of propoſitions about multiples; and that without any neceſſity, for the firſt reference to it is in the 22d, and therefore the proper place of it is between the 19th and 20th: and the propoſition which is made a corollary to it in the Greek, appears to have been originally in the ſame place, for it is not uſed before the 20th; though, if it had been given before, ſeveral of the former demonſtrations could have been ſhortened by it. It would be proper, therefore, to reſtore theſe propoſitions to their original ſituation; but this cannot be conveniently done with the fourth, becauſe geometers very often cite the propoſitions of the fifth book; and therefore the order is ſtill preſerved, but the demonſtration is rendered as full, and its dependence on the fifth definition as plain as poſſible.

Likewiſe, particular equimultiples are taken of the ſecond and fourth, becauſe it is thought, that the reader will better perceive the force of the demonſtration, when he ſees the equimultiples of the ſecond and fourth, which contain them the ſame number of times that the equimultiples of the firſt and third contain them. And the ſame particularity is uſed in all the following propoſitions, where the demonſtrations require equimultiples of the ſecond and fourth to be taken. But the demonſtrations could have been made in the ſame way, though any equimultiples had been taken of them; and Euclid's demonſtrations could have been conducted the ſame way, if he had taken equimultiples in the manner that is now done, and they would have been as general as when general equimultiples are taken, for the generality of the demonſtration depends wholly upon the general equimultiples of the firſt and third.

The 7th would have anſwered the purpoſe better, if it had been made more general: Thus, "If the firſt of four magnitudes be equal to the third, and the ſecond to the fourth; the firſt is to the ſecond, as the third to the fourth:" and we find that Euclid underſtood it in this general ſenſe, for the firſt application of it which he makes, is to four magnitudes in the 15th of this book. But however it be expreſſed, it is evident, from the nature of equal things, that in all caſes relating to their magnitude only, one of them may be ſubſtituted for another.

There are ſix propoſitions added to this book, of which the firſt five are placed between the 19th and 20th, becauſe ſome of them are neceſſary to the following propoſitions. It was ſhewn, in the notes on the 4th, that this is the proper place of the ſecond of them, and it is evidently the proper place of the fifth; and it was thought to be the beſt way to place them altogether.

The ſixth of them is one of thoſe relating to compound ratio, which Dr Simſon placed at the end of this book, and it is thought to be ſufficient for ſhewing, that what is ſaid of compound ratio by geometers, may be underſtood from the 22d and 23d of this book.

BOOK VI.

Book VI.

THE 2d definition is made more general and accurate, than in the former editions of Euclid.

In the enunciation of the 7th propoſition, the limitation is altered, ſo as to agree with what is proved in the demonſtration to be the caſe, when the triangles are not equiangular, by which means a diſtinction of caſes is avoided. But it is as eaſily known, when the preſent exception does not take place, as when the former does not take place.

Dr Simſon obſerves, that the 18th is vitiated, for it is only demonſtrated of quadrilaterals: and there certainly is but one caſe mentioned in the Greek, which is conſtructed and demonſtrated in a manner very different from Euclid's. But when a perſon, acquainted with Euclid's manner, attempts to reſtore it, he naturally falls into two caſes, that of a triangle, and that of a quadrilateral; for in the Greek, the caſe of the triangle is firſt conſtructed and demonſtrated, and then the quadrilateral is conſtructed from it. Theſe two caſes are now diſtinguiſhed; and from them it appears how to extend it to figures of five or more ſides, as Dr Simſon has done.

Book VI.

Many complaints have been made of the 27th, 28th, and 29th; but they are not in reality more difficult than the 25th, of which there are no complaints. The enunciations, however, are very obſcurely expreſſed; and the conſtructions of the 28th and 29th are not ſufficiently full, for it is taken for granted, that the reader can make a parallelogram ſimilar to one given, and equal to the ſum or difference of two rectilineal figures, though the method of finding their ſum or difference has not been particularly pointed out. Theſe are defects which ought undoubtedly to be removed; eſpecially as theſe two problems are the moſt uſeful of all in the Elements. It is therefore thought proper to alter the enunciations, ſo as to preſerve the ſame meaning, and to conſtruct the 28th and 29th more fully, in conſequence of which the demonſtrations are a little ſhortened. In the 28th, the parallelogram AP is ſaid by the ancients, to be applied to AB, deficient by the parallelogram PB, which is evidently ſimilar to the given parallelogram EF. And in the 28th, the parallelogram AX is ſaid to be applied to AB, exceeding by the parallelogram BX, which is ſimilar to the given one EL.

The caſe of Prop. D, which was omitted in the former editions, is now inſerted. And Prop. E is added on account of its great uſe in geometry.

BOOK XI.

DEF. VI.

Book XI.

IT is impoſſible to demonſtrate the properties and relations of ſolids contained by plane figures, without taking into conſideration the angles which theſe planes make with one another; becauſe their magnitude depends upon the angles made by their planes, as well as upon the magnitude of their plane figures. But in the propoſitions concerning ſolids, in this book, there is no mention made of theſe angles. In the 25th, parallelopipeds are ſaid to be equal, when they are contained by equal and ſimilar parallelograms: and in the 28th the priſms are ſaid to be equal, becauſe they are contained by the ſame number of equal and ſimilar planes. But ſolids may be contained by the ſame number of equal and ſimilar plane figures, and not be equal to one another, as ſhall be made manifeſt in the notes on the 10th definition, from Dr Simſon: It is alſo neceſſary to their equality, that their equal and ſimilar planes make equal angles with one another: and the acknowledged accuracy of Euclid obliges us to believe, that this laſt condition was not overlooked by him. And the 6th and 7th definitions muſt have been inſerted for this very purpoſe, for they are of no other uſe in this book. But in the 6th definition, the inclination of two planes is ſaid to be an acute angle, whereas theſe angles may be right or obtuſe angles; and therefore the word inclination, according to this defintion, is too limited to denote theſe angles. In the caſe of obtuſe angles, we may produce one of the ſides of the obtuſe angle, and thus obtain an acute angle on the other ſide of one of the planes: but this would oblige us to introduce a condition into the propoſitions where the term is uſed, which is not otherwiſe neceſſary; and would very much embarraſs the demonſtrations: as will be evident, if we conſider what would be the caſe in plane figures, if by the word angle were meant only an acute angle. Thus, in the fourth propoſition of the firſt book, not only muſt the angles contained by the equal ſides be equal, but theſe equal acute angles muſt be either both within the triangles, or elſe both without them, otherwiſe the propoſition is not true; and the ſame or greater embarraſſments muſt be introduced into the other propoſitions in which angles are concerned; and therefore the term angle, cannot be uſed in this limited ſenſe; but either its meaning muſt be extended, or a new term invented. For the ſame reaſon, we cannot uſe the word inclination in the limited ſenſe given to it in this definition: nor can it be eaſily believed, that Euclid defined the inclination of planes, by an acute angle, for he uſes the word κλισις in as extenſive a meaning as the word γωνια, in the 8th definition of the firſt book; and there is the ſame neceſſity for doing ſo in this definition. It is therefore certain, that the word acute ought not to be in this definition; and as it might very readily have been inſerted in it, from the preceding definition, through the inattention of ſome tranſcriber, this is probably the way that it has come to be in it. It is therefore now thrown out, by which means the term inclination is rendered as extenſively uſeful in ſolids, as the term angle is in plane figures.

DEF. IX.

This definition has fewer conditions than what are neceſſary, for ſolids may be contained by the ſame number of ſimilar planes, and not be ſimilar, as Dr Simſon obſerves. The ſimilarity of plane figures is defined from the proportionality of their ſides, and the equality of the angles contained by them; and in the

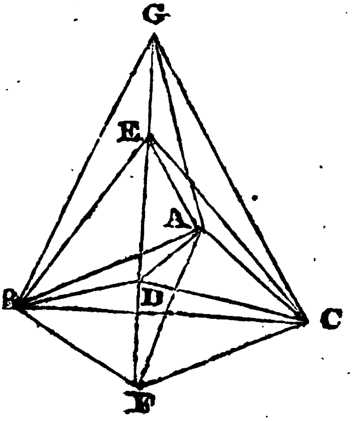
Book XI. ſame manner, the ſimilarity of ſolids ſhould be defined from the ſimilarity of their containing planes, and the equality of the angles made by theſe planes. To this laſt condition we are naturally led by the order of the definitions; for Euclid muſt have inſerted the 6th and 7th definitions, for the very purpoſe of determining the ſimilarity of ſolids, becauſe they can be of no other uſe in the doctrine of ſolids, and he certainly would not have given them without ſome deſign of uſing them. Dr Simſon thought that the condition wanting was the equality of their ſolid angles, and therefore he changed the order of the definitions: but we have not ſufficient reaſon for thinking this; and the equality of ſolid angles cannot be proved, without going far away from the text of Euclid, unleſs the plane angles containing them be diſpoſed in the ſame order and ſituation; which is a condition not neceſſary to the ſimilarity of ſolids. Beſides, the equality of the ſolid angles is of no uſe in this matter, but for determining the like inclinations of the planes, which can be better done without it, as ſhall be ſhown in the notes on Prop. A, B, and C, and the 26th propoſition of this book. The order of the definitions and the condition wanting are now reſtored.

DEF. X.

This definition is, that "Similar and equal ſolids are thoſe that are contained by the ſame number of ſimilar and equal planes." Now this is not a definition, but a theorem, and it is not always true, as Dr Simſon has ſhown by the following example.

"Let there be any plane rectilineal figure, as the triangle
a 12. 11. ABC, and from a point D within it, draw [a] DE at right angles
to the plane ABC; take DE, DF equal to one another, upon
oppoſite ſides of the plane; and let G be any point in EF; join
DA, DB, DC; EA, EB, EC; FA, FB, FC; GA, GB, GC:
Becauſe the ſtraight line EDF is at right angles to the plane
ABC, it makes right angles with DA, DB, DC, which it
meets in that plane; and in the triangles EDB, FDB, ED and
DB are equal to FD and DB, each to each, and they contain
b 4. 1. right angles; therefore the baſe EB is equal [b] to the baſe FB:
In the ſame manner, EA is equal to FA, and EC to FC: and
in the triangles EBA, FBA, EB, BA are equal to FB, BA, and
the baſe EA is equal to FA; wherefore the angle EBA is equal
c 8. 1. [c] to FBA, and the triangle EBA equal [b] to the triangle FBA,
and the other angles equal to the other angles; therefore theſe
d 4. 6. & 1. triangles are ſimilar [d]. In the ſame manner, the triangle EBC
Def. 6. is ſimilar to FBC, and EAC to FAC; therefore there are two
ſolid

ſolid figures, each of which is contained by ſix triangles ; one of them by three triangles, the common vertex of which is G, and their baſes the ſtraight lines AB, BC, CA ; and by three other triangles, the common vertex of which is the point E, and their baſes the ſame lines AB, BC, CA : the other ſolid is contained by the ſame three triangles, the common vertex of which is G, and their baſes AB, BC, CA, and by three other triangles of which the common vertex is F, and their baſes the ſame ſtraight lines AB, BC, CA: Now the three triangles GAB, GBC, GCA are common to both ſolids, and the three others EAB, EBC, ECA of the firſt ſolid, have been ſhewn equal and ſimilar to the three others FAB, FBC, FCA of the other ſolid, each to each ; therefore theſe two ſolids are contained by the ſame number of equal and ſimilar planes : But that they are not equal, is manifeſt, becauſe the firſt of them is contained in the other ; therefore it is not univerſally true, that ſolids are equal which are contained by the ſame number of equal and ſimilar planes." It is likewiſe manifeſt, that theſe ſolids are not ſimilar, ſince their ſimilar planes have not the ſame inclination to one another,



DEF. XI.

It is neceſſary to add a condition to this definition, becauſe Euclid underſtood his ſolid angles to be ſuch as had the inclinations of all the planes containing them inward, or towards the angle, as is manifeſt from the 21ſt prop. of this book : for it is there proved, that every ſolid angle is contained by plane angles, which together are leſs than four right angles, which would not always be the caſe, if the inclinations of ſome of the planes were inwards, and thoſe of others outwards ; as was ſhewn by M. le Sage, in a paper given in to the Academy of Sciences at Paris, in the year 1756.

Book XI.

PROP. VII. XIII. XXXVIII.

Dr Simson ſhows, that the firſt and laſt of theſe propoſitions have been put into the Elements, by ſome unſkilful editor, becauſe they are evident from the definitions of the 1ſt and 11th books. And the 13th is fully as evident, for it was demonſtrated in the 6th of this book, that the ſtraight lines mentioned in this 13th, as perpendiculars to the ſame plane, muſt be parallel, and therefore they cannot paſs through the ſame point. Theſe propoſitions are retained for no other reaſon, but to preſerve the number of the propoſitions: and ſhort demonſtrations are given of them.

PROP. VIII.

It is proved, that the angles ABD, ADE are right angles in the ſame words, both in this and the ſixth propoſitions: This repetition is now left out of the 8th, and aſſumed from the ſixth.

PROP. XVI.

In the Greek, this propoſition is demonſtrated indirectly; but a direct demonſtration is now given on the ſame principles, which makes it ſomething ſhorter.

PROP. XXIII.

This propoſition is much ſhortened, by firſt demonſtrating, that AP is greater than the greateſt ſide of the triangle LMN; after which, the demonſtration is nearly as in the firſt caſe of the former editions. This and the 22d, on which it depends, are of no uſe in the preſent ſtate of the Elements, either for underſtanding the nature of ſolid angles, or for conſtructing any of the following propoſitions; they may therefore be omitted without any loſs.

Beſides, unleſs the given angles be equal, there may be two different ſolid angles conſtructed as in the propoſition, each of them contained by plane angles, equal to the given ones; but placed in a different order. Thus, if MN be not equal to LN, and another triangle be deſcribed on the ſame ſide of ML, with the triangle LMN, ſo that the ſide of it terminated at M be equal to LN, and the ſide terminated at L equal to MN; and from its third angle, a ſtraight line be drawn to R; there will be another ſolid angle deſcribed at R, contained by three plane angles, equal to the given ones: But though theſe two ſolid angles have the angle LRM common, and the remaining angles equal, each to each; they cannot be made to coincide, becauſe their plane angles are placed in a contrary order; and therefore they cannot be proved to be equal by the help of any of the propoſitions at preſent in the Elements. And this obſtacle prevents us from receiving any aſſiſtance from this propoſition in filling up the chaſm which is evident in this part of the Elements: for the propoſitions before the 25th are not ſufficient for demonſtrating thoſe that follow; and therefore it is neceſſary to inſert other propoſitions for that purpoſe.

PROP. A.

This propoſition of Dr Simſon is very uſeful for enabling us to reaſon about ſolid angles: but his demonſtration can only be applied, when the plane angles which contain the ſolid ones, are none of them right angles; for if one of them, as CAD, be a right angle, the ſtraight line LK, which is at right angles to AC, does not meet AD, in which caſe his demonſtration fails: and this is a caſe that occurs very frequently; it is therefore neceſſary to change his demonſtration into a more general one.

A corollary is alſo added, which is as uſeful as the propoſition, and is neceſſary to the demonſtration which is now given of the 28th.

PROP. B. & C.

Dr Simson has proved in his Prop. B, that ſolid angles are equal, which are contained by three plane angles, equal to one another, each to each, and diſpoſed in the ſame ſituation and order. But if they be in a contrary order, his demonſtration cannot be applied: and this is a caſe that occurs as frequently as the other. But if he had demonſtrated both caſes, it would not have anſwered the purpoſe he intended by it, unleſs he had alſo proved, that whenever two ſolid angles, contained by the ſame number of plane angles equal each to each, are equal to one another, the plane angles have the ſame inclination to one another; for the ſimilarity and equality of ſolids do not depend upon the equality of their ſolid angles, but upon the like inclination of their planes: and thus, Dr Simſon, in demonſtrating his Prop. C. is obliged to conclude, that the ſolid angle at A, coincides with the ſolid angle at K, becauſe they are equal, which is not a legitimate inference, for ſolid angles may be equal, though they cannot be made to coincide: But though it were, it ought not to have been aſſumed without proof. So that, though

Dr Simſon cited Prop. B, in ſeveral propoſitions about ſolids, the ſteps for proof of which he cited it, do not depend upon it; but on its converſe, or rather on the like inclination of the containing planes. This Prop. B, of Dr Simſon, is therefore of no uſe in the doctrine of ſolids. It is accordingly left out, and his Prop. C is now made Prop. B.

Again, Of this propoſition, it is to be obſerved, that there is a limitation in it, expreſſed by the words, alike ſituated, which confines it to the half of the ſolids contained by equal and ſimilar planes having the ſame inclination to one another. When a plane in one of the ſolids is placed in the ſame ſituation with the plane which is equal and ſimilar to it in the other ſolid, it is neceſſary, that the other ſimilar planes be on the ſame ſide of theſe ſimilarly ſituated planes, and likewiſe be diſpoſed in the ſame order, in the two ſolids. Thus, when the plane figure AC is made to coincide with the plane figure KM, which is ſimilar and equal to it, the ſimilar plane figures AF and KP muſt be on the ſame ſide of the common plane KM, as well as have the ſame inclination to it, for if they be on different ſides of it, they cannot be ſaid to be alike ſituated, in reſpect of that common plane: and thus, Dr Simſon explains the words alike ſituated, by "the planes being diſpoſed in the ſame ſituation and order," in his notes on the 25th and 26th of this book.

But with this limitation, the propoſition is not confined to ſolids, which have none of their ſolid angles contained by more than three plane angles, but it is applicable to all ſolids of which the equal and ſimilar planes have the ſame inclination to one another: and as this extenſion of the propoſition is neceſſary, in order to demonſtrate the equality of ſolids which have not their equal and ſimilar planes alike ſituated, the enunciation is now expreſſed more generally than before.

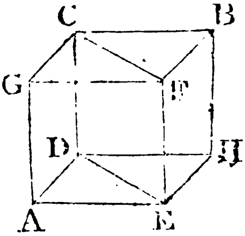
That which is now made Prop. C, is required in the demonſtration of the 28th: and it may be improved ſo as to demonſtrate the equality of ſuch ſolids as are excluded by the limitation of Prop. B.

PROP. XXVI.

It was neceſſary to change the enunciation of this propoſition, ſo as to agree with the demonſtration. In the former enunciation, it was propoſed to make a ſolid angle equal to a given one; but in the demonſtration, the plane angles only which contain the ſolid angles, are proved to be equal; and this is not always ſufficient for determining the ſolid angles to be equal, as Dr Simſon has proved: and it is certain, that Euclid introduced this propoſition only for the purpoſe of conſtructing the 27th and the 36th by it; and in both theſe, the plane angles only are required to be made equal, though the Greek editor calls this making the ſolid angles equal. Dr Simſon changed the expreſſion in the 36th, and Commandine changed it in the 27th, in the ſame manner that the enunciation of the 26th is now changed. Shorter demonſtrations are now given of the 26th, and the 35th, by the help of Prop. A, which is a propoſition of the moſt extenſive uſe in the contemplation of ſolid angles.

PROP. XXVIII.

In this book, we have a remarkable inſtance, how eaſily even the greateſt geometers may be deceived in geometrical matters, and how liable we are to take things for granted, upon the authority of others. From Theon's time to that of Dr. Simſon, all the propoſitions concerning the equality of ſolids were inſufficiently demonſtrated: inſtead of the ſolids being proved equal or ſimilar, it was only proved, that the planes containing them were equal or ſimilar. Dr Simſon reſtored it in a great meaſure to its original purity; and the doctrine of parallelopipeds has ſince ſtood upon a firm baſis, except in ſo far as it was made to depend upon his Prop. B. But he inadvertently overlooked this 28th propoſition, and did not take notice, that the planes of the two priſms to be proved equal, are not alike ſituated: and thus, the doctrine of priſms, pyramids, cylinders, cones, and ſpheres, ſtand upon the ſame infirm foundation as before, for all theſe depend upon this 28th, and are connected with the doctrine of parallelopipeds, by means of it, and the corollary to the 32d, which depends upon it. But the 28th is not ſufficiently demonſtrated by the help of Prop. B, or what Dr Simſon calls Prop. C, for it is required in that propoſition, that the planes of the ſolids be alike ſituated, in reſpect of one another, and have the ſame inclination to one another: But in the 28th, the triangles GCF, CFB, which are proved to be equal and ſimilar, are not alike ſituated, in reſpect of the equal and ſimilar parallelograms GE, CH, when they are not rectangles: For the angle CBH is equal to GFE, becauſe CB, BH are parallel to GF, FE; and the angle BCD is equal to FGA, becauſe BC, CD are parallel to FG, GA: But the angle at B of the triangle BCF is not equal to the angle CFG of the triangle GCF; therefore the triangles are not alike ſituated in reſpect of the parallelograms; and on that account, they cannot



be made to coincide; for if the parallelogram CH be applied to GE, ſo that B be on F, and BC on FG, the triangles BCF, CGF fall on different ſides of the common plane GE: and the ſame is the caſe when B is applied to A, and BC to AE; for if the parallelograms be thus made to coincide, the triangles BCF, ADE fall on different ſides of the common plane GE. In order to prove the priſms to be equal by Prop. B, it is neceſſary that the angle CBH be equal to the angle AGF, and FEH to CGA, which can never happen, except each of them be a right angle; for AGF, GFE, that is, AGF, CBH are together equal to two right angles, and therefore, if one of them be leſs, the other muſt be greater than a right angle: Wherefore this propoſition is not ſufficiently demonſtrated in the former editions. A new demonſtration is now given by the help of Prop. C, by which means we are enabled to compare together all ſolids, though contained by planes not alike ſituated in reſpect of one another.

PROP. XXIX. XXXI. XXXIII.

A MORE general demonſtration is given of the 29th: and the figures of the 31ſt and 33d are made more ſimple, by means of which the demonſtration of the latter is ſhortened. Profeſſor Playfair, in his demonſtration of this 33d, ſuppoſes, that ſimilar parallelopipeds have to one another the ratio which is compounded of the ratios of their baſes, and of their inſiſting lines; but as he has not before proved, that the inſiſting lines are to one another as the altitudes, the ſuppoſition is unwarrantable.

PROP. XXXIV.

THE demonſtration of this is now much ſhortened, by making the conſtruction of the whole ſimilar to what was before given of the ſecond part of the firſt caſe; and by demonſtrating the two parts of the propoſition ſeparately.

PROP. XXXV.

THE only uſe of this propoſition in the Elements is, to demonſtrate the following corollary by it; and the corollary is uſed in the following propoſition. But it is beſt to demonſtrate the corollary firſt, becauſe it is the only part that is uſed, and its demonſtration is ſhorter than the other; and that of the propoſition may be eaſily made from it. This method is now followed, and the demonſtration is made ſhorter by the help of Prop. A.

BOOK XII.

IN the enunciation of the ſecond propoſition, it is ſuppoſed poſſible for ſome ſtraight line to be equal to the circumference of a circle; and if the axiom be granted, this is alſo evident; for it follows from the axiom, that the perimeter of any circumſcribed polygon is greater than the circumference, and that the perimeter of any inſcribed polygon is leſs than it; therefore there muſt be ſome ſtraight line leſs than the former, and greater than the latter, that is neither greater nor leſs than the circumference, and conſequently is equal to it.

It is likewiſe ſaid, in the conſtruction of this 2d, that if DF be greater than the circle, there is ſome rectangle DH equal to the circle. It would not alter the demonſtration, though DH be taken greater than the circle; and this is ſhown to be poſſible in the firſt Prop. For if EF be biſected by a ſtraight line parallel to ED, it will cut off the half of the rectangle DF; and in the ſame manner may the half of the remainder be cut off; and ſo on; therefore, if this be done continually, there ſhall at length remain a rectangle leſs than the exceſs of DF, above the circle: let this be the rectangle HG; therefore the rectangle DH is greater than the circle. But it is ſuppoſed, in the propoſition, that there is ſome rectangle equal to the circle; and in the demonſtration it is proved, that it can neither be greater nor leſs than DF; therefore DH may be taken any rectangle whatever leſs than DF; and then the import of the firſt part of the demonſtration is to inquire, what would follow from ſuppoſing it equal to the circle: and the abſurd conſequence which we neceſſarily draw from it, obliges us to relinquiſh the ſuppoſition, that any rectangle leſs than DF is equal to the circle, or that DF is greater than the circle. So that it is not neceſſary to accuracy, that we be able to determine the magnitude of DH, be a conſtruction; it is ſufficient if its being equal to the circle, be a ſuppoſition that is evidently poſſible. And the like is to be obſerved in the following propoſitions. In indirect demonſtrations, actual conſtructions may be often ſpared, in caſes where they are neceſſary in direct ones. Thus, in the 6th of the 1ſt Book, it evidently follows, from our ſuppoſing AB greater than AC, that ſome part of AB is equal to AC; and this part, whatever be its magnitude in reſpect of AB, may be called BD. And it ſeems to be uſeleſs labour to attempt the conſtruction of a line or figure which we intend to prove impoſſible.

LOGAR.

IN Trigonometry, the principles are explained fully, and the nature and uſe of the Tables of Natural Sines, Tangents, and Secants are ſhewn, as well as their conſtruction. But as it is uſual to perform the operations by Logarithms, their nature ſhall be explained here.

LOGARITHMS.

A SERIES of numbers, which increaſe by the ſame difference, is called an Arithmetical Progreſſion: and a ſeries of numbers, which increaſe by a common multiplier, is called a Geometrical Progreſſion.

Let the firſt term of an arithmetical progreſſion be placed above the firſt of a geometrical one, and the reſt in order, thus:

Arithmetical progreſſion,	0,	1,	2,	3,	4,	5,	6,	&c.
Geometrical progreſſion,	1,	4,	16,	64,	256,	1024,	4096,	&c.

Then it is evident, that if we are to multiply any term of the geometrical ſeries, as 64, by another term, as 16, the product can be found by adding the correſponding terms of the arithmetical ſeries, viz. 2 and 3, for the ſum 5 is the term correſponding to the product 1024: Thus, the uſe of a Table of arithmetical progreſſionals correſponding to a geometrical progreſſion, is evident; but in order to this, the terms muſt be increaſed in both, ſo that the geometrical may contain all the natural numbers of arithmetic. This may be done, by finding geometrical mean proportionals between the terms of the geometrical progreſſion, and as many arithmetical means between the terms of the arithmetical one. Now, an arithmetical mean is half the ſum of the adjacent terms, and a geometrical mean is the ſquare-root of the product of the adjacent terms. If this be done, the ſeries will contain twice as many terms as before, and will become

Arithm. prog.	0,	½,	1,	1½,	2,	2½,	3,	3½,	4,	4½,	5,	5½,	6,	&c.
Geom. prog.	1,	2,	4,	8,	16,	32,	64,	128,	256,	512,	1024,	2048,	4096,	&c.

Theſe progreſſions may be interpolated in the ſame manner, by new terms; and the operation may be continued until all the natural numbers occur in the geometrical progreſſion. And then the terms of the arithmetical ſeries correſponding to theſe numbers, are called the Logarithms of them.

Hence, Logarithms are artificial numbers, by the aſſiſtance of which, addition ſupplies the place of multiplication, and ſubtraction of diviſion.

In the common tables of logarithms, the progreſſions are
Arithmetical. 0, 1, 2, 3, 4, 5, &c.
Geometrical. 1, 10, 100, 1000, 10,000, 100,000, &c.
having terms interpolated continually, as was ſhewn in the former ſeries, until all the natural numbers are in the geometrical ſeries, and the numbers in the table of logarithms are thoſe of the arithmetical ſeries, which correſpond with the natural numbers in the geometrical ſeries.

Hence, the logarithms of all numbers between 1 and 10, are fractions; the logarithms of all numbers between 10 and 100, are mixed numbers that have 1 for the integer, and thoſe of numbers between 100 and 1000, have 2 for the integer; and ſo on: That is, the units in the integer are always leſs, by one, than the places in the correſponding number. The integer is called the Index, becauſe it ſhews the number of figures in the anſwering number.

TO find the Logarithm of any Number from the Tables.

LOOK for the three higheſt figures in the margin on the left ſide, and in that line, in the column which has the fourth figure at the top, you will find the logarithm for theſe four figures. If the number have more figures in it, take the difference between this logarithm, and the next greater, and multiply it by the remaining figures, and from the product cut off as many figures as are in the multiplier, the reſt added to the logarithm for the firſt four figures, give the logarithm required. The index is always one leſs than the number of integers.

TO find the Number for a given Logarithm.

IF the logarithm be found in the table, the three firſt figures are on the ſame line in the margin, and the fourth is at the top of the column. But if the logarithm be not in the table, take the number anſwering the next leſs logarithm, and ſubtract this leſs logarithm from the given one, and alſo from the next greater; then annexing cyphers to the firſt remainder, divide it by the other, to get the fifth, ſixth, &c. figures. The integers muſt be one more than the index, and the reſt are decimals.

Thus, To find the logarithm of 73284, on the line that has 732 in the margin, and in the column that has 8 at the top, there is found 864985, the logarithm of 7328, and the difference between it and the next logarithm is 60, which, multiplied by 4, gives 240; therefore, adding 24 to .864985, we have 4.865009 for the logarithm of 73284, with 4 for an index, becauſe the number has five places.

Again, To find the number anſwering to the logarithm 4.597179. Look for the next leſs logarithm, and the number anſwering to it is 3955, and the difference between it and the given logarithm is 33, and the difference between it and the next greater in the table is 110; therefore divide 330 by 110, and the quotient 3, annexed to 3955, gives 39553 for the number anſwering the given logarithm.

OF the Tables of Sines, Tangents, and Secants.

These tables have the degrees at the top, and the minutes on the left ſide of the page, when the degrees are not above 45°; but if they be greater than 45°, the degrees are at the foot, and the minutes on the right ſide.

If the logarithms of the numbers in theſe tables be taken, and arranged in the ſame manner in tables, they form the tables of Artificial Sines, Tangents, and Secants, which ſupply the place of the natural ones, in the ſame manner that logarithms ſupply the place of natural numbers.

TO find the Artificial Sine of 37° 23′ 12″.

Look for the page that has 37° at the top, and there, on the line that has 23′ on the left ſide, and in the column titled ſine, is 9.7832922, the ſine of 37° 23′: and the difference between it and that of 37° 24′, is 1653. Then, as 60″ to 12″, ſo is 1653 to 331, the proportional difference for 12″, which, added to 9.7832922, gives 9.7833253 for the ſine of 37° 23′ 12″.

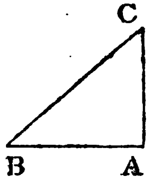
Again, To find the degrees, and parts of a degree, of which 10.2738462 is the artificial tangent.

Look for the neareſt tangent, 10.2737163, and becauſe it is titled Tang. at the foot, take the degrees at the foot, and the minutes on the right, and it gives 61° 58′. And the difference between this tangent and the one above it is 3046, and the difference between it and the given one is 1299; therefore, as 3046 to 1299, ſo is 60″ to 26″; ſo that 10.2738462 is the tangent of 61°, 58′, 26″.

To

TO ſhew the uſe of Logarithms in Trigonometry.

Let ABC be a plane triangle, right angled at A, of which the hypothenuſe BC is 368 feet, and the angle at B 42° 35′, and let the ſides AC and AB be ſought.



By the 2d Cor. to 2d Prop. of Plane Trig. R, is to the ſin. B, as BC to CA: conſequently, from the nature of numbers, CA is found by multiplying BC by the ſin. B, and dividing the product by R; and this is done, by adding the logarithm of BC, and the artificial ſine of B together, and ſubtracting the artificial radius from the ſum, for the remainder will be the logarithm of CA. And in the ſame manner may BA be found.

To find CA.		To find BA.	
As the radius	10.0000000	As R.	10.0000000
To ſin. B 42° 35′	9.8303717	To cos. B. 42° 35′	9.8670512
So is BC 368 f.	2.5658478	So is CB. 368	2.5658478
	12.3962195		12.4328990
	10.0000000		10.0000000
To CA 249.01 f.	2.3962195	To BA. 270.96	2.4328990

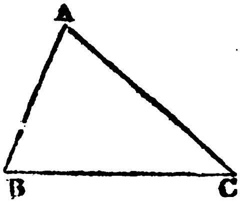
When the firſt term is not radius, inſtead of ſubtracting its logarithm, we often add what it wants of the radius to the other two, and take 10 from the index of the ſum; this want is eaſily got, by ſubtracting the right-hand figure of the logarithm from 10, and all the reſt from 9; and it is called the Arithmetical Complement of the Logarithm.

And becauſe the ſquare of a number is obtained by adding its logarithm to itſelf, or by doubling it; the ſquare root of a number will be got by dividing its logarithm by 2.

To illuſtrate this, let the three ſides of a plane triangle ABC be, AB 228, AC 136, and BC 318 feet; and let the angle at A be required.

By the 8th Prop. of Plane Trig. the rectangle contained by AB, AC is to the rectangle contained by $\frac{1}{2}$ the ſum of the three ſides and its exceſs above BC, as the ſquare of R to the ſquare of the coſine of $\frac{1}{2}$ A. And the rectangles are got by adding their logarithms, and the ſquares by doubling them. Therefore, add the three ſides together, and from $\frac{1}{2}$ the ſum ſubtract BC. Then add into one ſum the Arithmetical complements of AB and AC, and the logarithms of the $\frac{1}{2}$ ſum and remainder, and $\frac{1}{2}$ the ſum of theſe four will be the coſine of $\frac{1}{2}$ the angle A.

AB	228	Arith. comp.	7.6420652
AC	136	Arith. comp.	7.8664611
BC	318		
	682		
Half ſum	341	Logarithm.	2.5327544
	318		
Remainder	23	Logarithm.	1.3617278
			2)19.4030085
Half the angle A	59° 43′ 22″	coſine	9.7015042
	2		
The angle A	118 36 44		



PRACTICAL GEOMETRY.

In order to determine the magnitudes of any kind of quantities, there is ſome quantity of the ſame kind aſſumed, as ſufficiently known, and the others are ſaid to be known, when their relation to it is known.

To avoid fractions, this aſſumed quantity is commonly among the leaſt that are in common uſe. But after it is aſſumed, any multiple or part of it may be aſſumed for the ſame purpoſe. Quantities thus aſſumed, are called Meaſures of other Quantities.

As meaſures are of general utility, it is often neceſſary to determine their form, or other circumſtances, by means of which their magnitude may be known.

This Treatiſe is divided into three parts. The firſt treats of Lines and Angles; the ſecond of Superficies; and the third of Solids.

PART I.

OF LINES AND ANGLES.

The leaſt meaſure of lines with us, is an inch, 12 of which make a foot; and 3 feet make a yard. But diſtances are often meaſured by a chain of 22 yards, of which 80 make a mile. If the length of a pendulum, vibrating ſeconds at London, be divided into 313 equal parts, eight of theſe parts will make an inch.

The Scots inch is a little longer than the Engliſh inch, for 185 Scots inches make 186 Engliſh inches. There are 37 Scots inches in an ell, and 24 ells in a Scots chain, 80 of which make a Scots mile; ſo that 55 Scots miles are equal to 62 Engliſh.

PROB. I. Fig. 1.

Plate I. TO deſcribe the conſtruction and uſe of the inſtruments uſed for taking angles.

Fig. 1. The Quadrant is the fourth part of a circle divided into ninety degrees, and, if the limb be large enough, each degree ſubdivided into quarters or minutes. To the radius which paſſes through the ninetieth degree, two ſights are adapted; and a thread with a plummet is hung from the centre. The uſe of it is to take angles in a vertical plane.

Fig. 2. The Theodolite is a circle divided into 360 degrees, with one or more indices fixed on its centre; theſe indices are fitted with Nonuis's diviſions, for finding the parts of a degree. The uſe of it is to take angles in an horizontal plane. It has a great deal of apparatus, in order to adjuſt it, ſuch as three ſtaffs, with joints and ſcrews, and a level, for ſupporting it and placing it horizontally; it has alſo a teleſcope for obſerving objects; and beſides theſe, it has a vertical arch for ſhewing the inclination of lines to the horizon, and a compaſs for ſhewing their inclination to the meridian.

PROB. II. Fig. 3.

Pl. I. Fig. 3. TO deſcribe the conſtruction of the Geometrical Square.

This is a ſquare with a line and plummet hanging from one of its angles A, and each of the ſides BE and ED is divided into 100 equal parts: and there are two ſights C and F fixed on the ſide AD.

There is alſo an index GH, with ſights, which, when there is occaſion, can be joined to the inſtrument, and made to move about the centre A.

PROB. III.

Pl. I. TO deſcribe the conſtruction and uſe of a line of chords, and of a line of equal parts.

It is often neceſſary to lay down a figure on paper, like to another figure, for which purpoſe, theſe lines are required, in order to make their angles equal, and their ſides proportionals.

Fig. 4. The line of chords is made thus: Let BAC be a quarter of a circle, and join BC, and divide the arch BC into 90 degrees: Then from C transfer the diſtances of the diviſions to the ſtraight line CB, and mark them with the degrees in the correſponding arch. The chord of 60 degrees being the ſide of an inſcribed hexagon, is equal to the radius of the circle, by cor. to the 15th of 4th B. of El.

Fig. 5. If now the angle EDF is to be meaſured, take the chord of 60°, and from the centre D deſcribe with that diſtance the arch FG. Then, if the diſtance FG applied on the line of chords, from C towards B, gives 25°, this ſhall be the meaſure of the angle propoſed.

When an obtuſe angle KDE is to be meaſured, produce KD to F, and meaſure its ſupplement FDE, and then KDE will be known.

Fig. 6. But if an angle of 50° is to be made at a given point M, in the line KL, from the centre M, with the diſtance MN, equal to the chord of 60°, deſcribe the arch NR. Take 50° from the line of chords, and make NR equal to it; and join MR; and it is plain, that NMR is an angle of 50°.

Fig. 7. The line of equal parts is made by taking any convenient diſtance for one of the parts, and laying it on the line, from one end to the other; after which, one of the parts is uſually ſubdivided into 10 equal parts. From ſuch a ſcale, any number leſs than 100, may be taken, by calling each of the greater diviſions 10, and each of the leſs diviſions one.

If upon one ſide of the line, thus divided, there be drawn ten others, at equal diſtances from one another, perpendiculars to the divided line, through the larger diviſions, will divide them all; and if one of the ſpaces between the perpendiculars of the two outermoſt lines, be each divided into ten equal parts, and from the end of one of them a line be drawn to the firſt diviſion of the other, and the reſt of the diviſions be joined in their order, a diagonal ſcale will be made, from which any number leſs than a thouſand may be taken.

Theſe lines, with ſeveral others, as lines of ſines, tangents, and ſecants, are uſually marked on the ſcales uſed by artiſts. But they are moſt convenient for uſe, when placed on a ſector, which is a jointed ſcale, like a carpenter's rule; for by means of it, an arch may be made of a given number of degrees to any radius, and diſtances may be taken greater or leſs, without altering their ratios; this is done, by opening the ſector, and extending the compaſſes from the number on one of the legs, to the ſame number on the other.

PROB.

PROB. IV.

Part I. Pl. I. Fig. 8.

TO lay down on paper, a rectilineal figure ſimilar to a given rectilineal figure.

For example, ſuppoſe the given figure a quadrilateral, of which one ſide is 235 feet, and the angle which it makes with the ſecond ſide 84°, and the ſecond ſide 288 feet, and the angle contained by it and the third ſide 72°, and the third ſide 294 feet. The part repreſenting a foot, is to be taken greater or leſs, according as you would have your figure greater or leſs. In this figure the 100th part of an inch is taken for a foot. Draw any ſtraight line AB, and take 235 feet from the ſcale, and place it from A to B, which determines its length; at B make, by Prob. 3. an angle ABC of 84°, and make BC 288 parts, in the ſame manner with AB. Then make the angle BCD 72°, and the ſide CD 294 parts, and join AD, and it will complete the figure. And the angles at A and D can be meaſured as in the laſt problem, and the ſide AD by the line of equal parts. In the ſame manner may a figure be drawn ſimilar to any given figure.

Cor. Hence, any problem in Plane Trigonometry may be reſolved by delineating the figure, as in this problem, and then meaſuring the unknown ſides and angles.

PROB. V.

Pl. I. Fig. 9.

TO deſcribe the manner in which angles are meaſured by the Quadrant.

Let the angle RAH, in a vertical plane, contained by the line AH parallel to the horizon, and AR coming from ſome remarkable point of a tower or hill, or from the ſun, moon, or a ſtar, be the angle to be meaſured. Hold the quadrant with the limb DCB downward, ſo that the eye at D may ſee the object R through the ſights in the ſide AD, and the degrees and minutes in the arch BC between the line and plummet, and B the fartheſt end of the limb will meaſure the angle RAH. For from the make of the quadrant, DAB is a right angle, therefore BAR is a right angle; and CAH is a right angle, for AC is perpendicular to the horizon; therefore, taking away the common angle BAH, the remaining angle HAR is equal to the remaining angle BAC, of which the meaſure is BC. Hence the angle DAC is equal to the angle RAZ.

To take an angle of depreſſion, as EAF. With the eye at A the centre of the quadrant, look through the ſights on the ſide AD to the object F. Then the arch BC is the meaſure of the angle EAF; for each of the angles CAE, BAF being right angles, take away the angle DAC, which is common, and the angle BAC is equal to EAF.

PROB. VI.

To meaſure an acceſſible height AB, by the ſquare. Pl. I. Fig. 10.

Let the inſtrument lean on a ſupport of a known height, and be directed towards A, ſo that the ſummet A may be ſeen through the ſights, in one of the ſides, as KL, and the plummet hanging freely, let it cut the ſide KN neareſt to the eye in N. Then, becauſe LN is parallel to AC, the angles KLN, KAC are equal [a], and LKN, ACK are right angles; therefore the triangles LKN, ACK are equiangular; and NK is to KL, as [b] KC to CA. Suppoſe the diſtance CK 96 feet, and KN 80 parts, then, as 80 is to 100, ſo is 96 to 120; which therefore is CA.

a 29. 1. E.
b 4. 6. E.

But if the obſerver be at G, ſuch a diſtance, that the plummet line paſſes through the angle P, oppoſite to H the centre; then, becauſe PG is equal to GH, GC is alſo equal to CA.

But let the diſtance BF or CQ be 300 feet, and let the plumline cut off 40 parts from the ſide SI oppoſite to the ſights. Then the angle SIZ is equal [a] to QZI, that is, to QAC: and ZSI, QCA are right angles, therefore the triangles ZSI, QAC are equiangular [c]; and ZS is to SI, as QC to CA; that is, as 100 to 40, ſo is 300 to 120; which therefore is the height of the object.

a 29. 1. E.
c 32. 1. E.

Note, If the height of a tower to be meaſured end in a point, as fig. 11. half the breadth or thickneſs BD of the tower is to be added to CD the diſtance of the obſerver from the tower, to get his diſtance from the foot of the perpendicular AB. Fig. 11.

PROB. VII.

To meaſure an acceſſible height by the quadrant. Pl. I. Fig. 12.

Find the angle C by the quadrant, as was directed in the 4th Prob. and meaſure CB. Then in the triangle ACB, right angled at B, are given CB, and the angle at C, to find BA: Wherefore, as R is to the tang. C, ſo is CB to BA.

PROB. VIII.

Part I. Pl. I. Fig. 13. TO meaſure an acceſſible height AB by means of a plane mirror.

Let the mirror be at C, in the horizontal plane BD; and let the obſerver go back to D, till he ſee the image of the ſummit in the mirror, at a certain point of it, which he muſt carefully mark; and let DE be the height of the obſerver's eye; then the angles DCE, ACB of incidence and reflection are equal, as is demonſtrated in optics; and CDE, CBA are right angles; wherefore, as CD to DE, ſo [a] is CB to BA.

Note, Inſtead of a mirror, the ſurface of water may be uſed.

[a] 4. 6. E.

PROB. IX.

Pl. I. Fig. 14. TO meaſure an acceſſible height AB by means of two ſtaffs.

Let the longer ſtaff DE be fixed perpendicularly in the ground, and move the ſhorter one FG backward, until the ſummit be ſeen over the ends F, D of the two ſtaffs: and let DC, FH be parallel to the horizon, or to GB. Then the angle DFH is equal [a] to ADC; and DHF, ACD are right angles; therefore FH is to HD, as [b] DC to CA; to which, if DE be added, the ſum is AB.

Fig. 15. Note, Many other methods might have been mentioned. For example, let EC be a ſtaff erected perpendicular to the horizon, and EF its ſhadow, and let BD be the ſhadow of the height AB: Then, AD, CF being parallel, the angle CFE is equal [a] to ADE; and CEF, ABD are right angles; therefore, as FE to EC, ſo is DB to BA [b]. And this is the caſe, if BD, EF be equally inclined to the horizon, as well as when they are parallel to it.

[a] 29. 1. E.

[b] 4. 6. E.

PROB. X.

TO meaſure an inacceſſible height AB by the Quadrant.

Chuſe the plane DC parallel to the horizon, and meaſure any diſtance DC in a ſtraight line with the perpendicular AB; and at C take the angle ACB, and at D take the angle ADC: and becauſe the angle ACB is equal to the angles CDA, DAC [a], the angle DAC is given; therefore, by the 3d Prop. of Pl. Trig. as the ſin. DAC to the ſin. ADC, ſo is DC to CA, which can

[a] 32. 1. E.

can therefore be found. And in the triangle ACB, right angled at B, by the 1ſt Propoſition of Plane Trigonometry, as R to ſin. ACB, ſo is CA to AB.

PROB. XI.

TO meaſure an inacceſſible height AB, by means of two ſtaffs. Pl. I. Fig. 17.

Let the obſervation be firſt made with the two ſtaffs DE and
FG, as in Prob. 8. then go off in a ſtraight line from the height
and firſt ſtation, to a ſecond ſtation, and there place the longer
ſtaff perpendicularly at RN, and then the ſhorter ſtaff at KO,
ſo that the ſummit A may be ſeen along their tops: Join KF,
ND, and draw NP parallel to AF; therefore the angle KNP is
equal [a] to KAF; and AKF is common to the triangles NKP, a 29. 1. E.
AKF; therefore, as KP : PN : : KF : FA [b]: but becauſe the tri- b 4. 6. E.
angles PNL, FAS are ſimilar, PN : NL : : FA : AS; therefore,
by equality [c], KP : NL : : KF : AS; therefore AS is found, to c 22. 5. E.
which add SB the height of the ſhorter ſtaff, and the ſum is the
inacceſſible height AB.

Note, In the ſame manner may an inacceſſible height be found by the ſquare, or a ſpeculum, or by the geometrical croſs.

PROB. XII.

FROM the top of a given height AB, to meaſure the diſtance BC. Pl. I. Fig. 18.

Let the angle of depreſſion DAC be taken, as in Prob. 4.
and becauſe AD is parallel to BC, the angle ACB is equal [a] to a 29. 1. E.
DAC; therefore, in the triangle ABC, right angled at B, are
given AB, and the angle C, to find BC; wherefore, as the tangent C : R : : AB : BC.

To do the ſame by the ſquare, place it ſo that C may be ſeen Pl. II.
through the ſights, and obſerve how many parts are cut off by Fig. 19.
the perpendicular: then the triangles AEF, ABC are ſimilar;
therefore, as EF : EA : : AB : BC [b]. b 4. 6. E.

PROB. XIII.

TO meaſure the diſtance of two places A and B, one of which A is acceſſible. Pl. II. Fig. 20.

Take any ſtation C, from which both A and B are ſeen, and meaſure AC. Place the theodolite at A, and direct the ſights to B, and then to C, and mark the degrees and parts of a degree on the limb between them, and they will be the meaſure of the angle BAC: take, in the ſame manner, the angle ACB; then the angle at B is known; therefore, by Prop. 3. Pl. Trig. ſin. B : ſin. C : : CA : AB, which will therefore be found.

Fig. 21. To do the ſame by the ſquare, place the ſquare horizontally at A, with the index on it, and direct the ſights of the ſquare along AB, and the ſights of the index along the other ſide of the ſquare, and obſerve through them ſome acceſſible mark C: then meaſure AC; and placing the ſquare at C, direct its ſights to A, and the ſights of the index to B. If the index cut the ſide RK adjacent to the ſights, as at K, then CR : RK : : CA : AB[a]. But if the index cut the ſide SL parallel to the ſights, as at L, then LS : SC : : CA : AG, the diſtance ſought, which will therefore be found.

[a] 4. 6. E.

Fig. 22. To do the ſame by four ſtaffs. Fix a ſtaff C, in the ſame ſtraight line with A and B, and draw the lines AH, CK parallel to one another, or perpendicular to CB; and place a ſtaff at any point H of AH, and find the point K of CK, which is in a ſtraight line with H and B, there fix a ſtaff, and meaſure AH, CK; and if HG be parallel to CA, the triangles KGH, HAB are ſimilar; therefore KG : GH or CA : : HA : AB.

Fig. 23. Note, A ſtraight line may be drawn in the field, perpendicular to another CH, from the point A, by making AB equal to AD, and fixing the ends of a line at B and D, the middle point of which has been previouſly marked; then let the line be ſtretched by the middle E, until both parts of it are equally ſtretched, and fix a ſtaff at E: and the ſtraight line AE is perpendicular to CH. In a manner ſimilar to this, may any problem that can be reſolved upon paper, by a ſcale and compaſſes, be done by ropes or cords in the field.

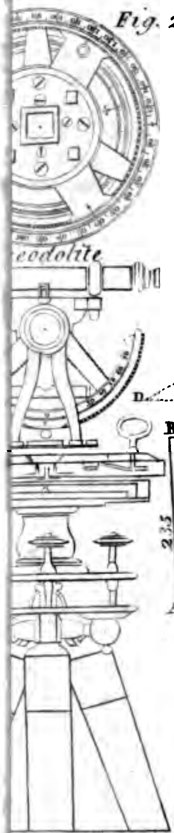
Fig. 24. A ſhort diſtance may be meaſured thus: Draw AC perpendicular to AB; and from C draw CD perpendicular to CB, meeting BA in D: Then DA : AC : : AC : AB. Or AC may be a ſtaff erected at A, and a maſon's ſquare being put on C, direct one of the ſides towards B, and mark the point D of the ground to which the other ſide is directed.

PROB. XIV.

TO meaſure the diſtance of two places, A and B, neither of which is acceſſible.

Chuſe

Fig. 2



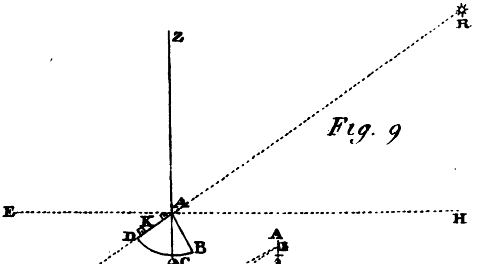


Fig. 9

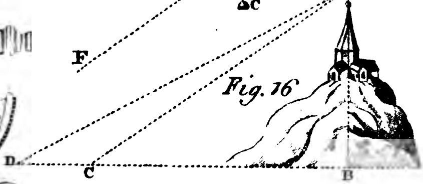


Fig. 16

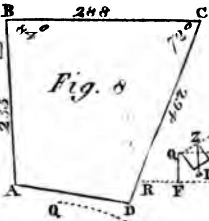


Fig. 8

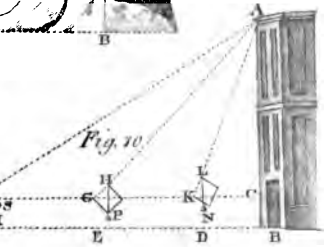


Fig. 10

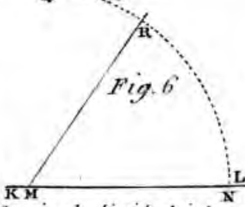


Fig. 6

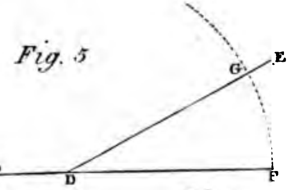
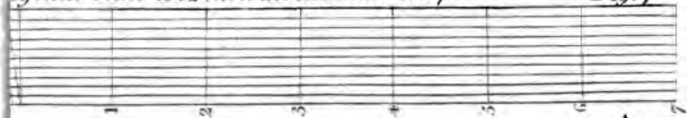


Fig. 5

agonal Scale of ½ inch divided into 100 parts

Fig. 7



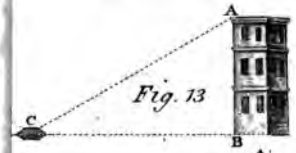


Fig. 13

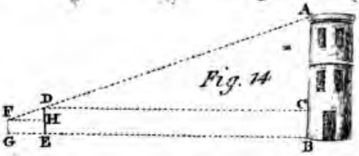


Fig. 14

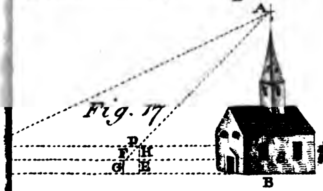


Fig. 17

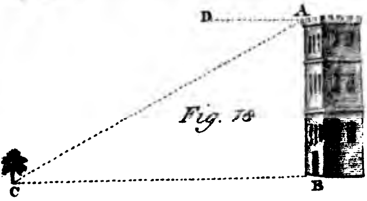


Fig. 18

Chuſe two ſtations C and D, from each of which, the places A and B may be ſeen; and let the angles ACB, BCD, and BDA, ADC be taken with the theodolite, and meaſure the diſtance CD, or find it by ſome of the preceding problems. Then, in the triangle ACD, are given the angles, and the ſide CD, to find AD, therefore ſin. CAD : ſin. ACD : : CD : DA. In like manner, in the triangle BCD, all the angles are given, and the ſide CD, to find DB; therefore ſin. CBD : ſin. BCD : : CD : DB. Then in the triangle ADB, are given AD and DB, and the angle ADB; therefore, by Prop. 4. Pl. Trig. the ſum of AD, DB is to their difference, as the tangent of $\frac{1}{2}$ the ſum of DAB, DBA to the tangent of $\frac{1}{2}$ their difference; whence theſe angles may be found, by Lem. 4. of Pl. Trig. and then ſin. ABD : ſin. ADB : : DA : AB, the diſtance to be found.

Note, It is not neceſſary that the points A, B, C, and D, be in one plane, but if they be not, the angles ACD, CDB muſt be taken with the theodolite.

PROB. XV.

Pl. II. Fig. 26.

TO meaſure a height AF placed on a ſteep, ſo that one can neither go near it, on a horizontal plane, nor recede from it.

Chuſe any two ſtations C and D, and find the angles ACD, and ADC, by the theodolite, and meaſure the diſtance CD: alſo with the quadrant, find the angles ACB and FCB. Then in the triangle ACD, the angles are known, and the ſide CD; therefore ſin. CAD : ſin. ADC : : DC : CA : and becauſe ACB is known, its complement CAB is known; and FCB is known, therefore ACF is known: Wherefore, all the angles of the triangle ACF are given, and the ſide CA, to find AF; and ſin. AFC : ſin. ACF : : CA : AF, the height required.

PROB. XVI.

Pl. II. Fig. 27.

TO find how much one place is more elevated than another.

Here a level and poles are uſed. Let the poles ab, cd, ef be erected perpendicular to the horizon. Then let the level be placed horizontally between the poles ab, and cd, and mark the points b and c of the poles, which are ſeen in a ſtraight line through the ſights of the level. Place in the ſame manner the level between the poles cd, and ef, and mark the points d and e

upon

upon them; and ſo on, until you arrive at the top A. Then meaſure ab, cd, ef, and add them together, and the ſum is the elevation of A above the place a; that is, it is equal to the height AB. Likewiſe the diſtances bc, de, fA, added together, give the horizontal diſtance of A from a; that is, they give aB.

COR. Hence the angle of elevation BaA, and the hypotenuſe aA may be found by Trigonometry.

Note 1. Sometimes the angles of elevation are taken from ſtation to ſtation, and the diſtances meaſured on the ground, by which means, between each ſtation there is formed a triangle, of which the hypotenuſe and the angle at the baſe are given, to find the baſe and perpendicular by Trigonometry.

Note 2. The level ought to be very accurate, and the ſights fitted with a teleſcope: and if the diſtance be great, an allowance muſt be made for the rotundity of the earth, at the rate of 7.96 inches for every mile meaſured on the earth; that is, the level at a mile's diſtance is 7.96 inches lower than what is found by the level.

Note 3. It is not neceſſary to go in a ſtraight line from one of the places to the other, but every two ſucceſſive ſtations may be taken in the moſt convenient direction.

PROB. XVII.

Pl. II.
Fig. 28.

TO find the diameter of the earth, from one obſervation.

Let a high hill AB be choſen near the ſea; and let the height of it be found as exactly as poſſible, by ſome of the foregoing methods. Then with a very exact quadrant, that can take an angle to ſeconds, find the angle ABE contained by the perpendicular AB, and the viſual line BE, which touches the earth at E: and let AF be perpendicular to AB. Then, in the triangle ABF, right angled at A, are given AB, and the angle ABF, to find AF and FB; and R : tan. B : : BA : AF, and R : ſec. B : : AB : BF; therefore AF and FB are found: and AF is equal[a] to FE, therefore BE is known. But the ſquare of BE is equal[b] to the rectangle AB, BD; and AB is given, therefore BD can be found; from which, ſubtracting BA, the remainder AD is the diameter of the earth.

[a] Cor. to 37. 3. E.

[b] 36. 3.

Note, There are many other methods of meaſuring the diameter of the earth, but they do not belong to this place: the lateſt attempts make the mean diameter to be 7913 Engliſh miles, though it was formerly determined, to be about 7970 miles.

OF SURVEYING LAND.

PART I.

IN ſurveying land, we uſe the chain for meaſuring lines; and in uſing it, there are 10 arrows, or long pins; and when the chain is ſtretched, the follower directs the leader into the ſtraight line which is to be meaſured, and there the leader fixes an arrow in the ground, and goes forward, and when the follower comes to the arrow, he again directs the leader into the ſtraight line to be meaſured, who then fixes another arrow in the ground, and proceeds, by which means they are enabled to meaſure any diſtance, without danger of miſtaking the number of chains. There is alſo an offset-ſtaff, 10 links long, and divided into 10 equal parts, for meaſuring the diſtances of the hedges, and other things from the ſtraight line meaſured by the chain; but care muſt be taken, to enter theſe ſmall diſtances in the field-book, with ſuch remarks as ſhall readily point out their ſituation, and diſtinguiſh them from one another.

The angles are to be taken with the theodolite; and likewiſe the elevation of the lines meaſured, when they are not parallel to the horizon, and then they ought to be reduced to horizontal lines, before they be entered in the field-book; or elſe their elevations ought to be ſo entered, that they may be eaſily known, and reduced afterwards, by the following

TABLE

For reducing Inclined Lines to Horizontal ones.

Deg. of elev.	Lin. Sub.	Deg. of elev.	Lin Sub.	Deg. of elev.	Lin. Sub.	Deg. of elev.	Lin. Sub	Deg. of elev.	Lin. Sub
4.05	¼	14.07	3	23.07	8	29.54	13	34.91	13
5.73	½	16.26	4	24.50	9	30.68	14	35.90	19
7.02	¾	18.20	5	25.84	10	31.79	15	36.87	20
8.11	1	19.95	6	27.13	11	32.86	16	37.81	21
11.48	2	21.56	7	28.36	12	33.90	17	38.74	22

Suppoſe the inclined line really to meaſure 1107 links, and the angle of elevation to be 19°. 95. Then looking in the Table againſt 19°. 95, I find 6 links, which multiplied by 11, is 66; and this ſubtracted from 1107, leaves 1041, the true horizontal length to be laid down in the plan.

After this, the field is to be delineated on paper, by a line of chords, and a line of equal parts: and the area is to be found by the problems in the ſecond part.

Small fields are very eaſily ſurveyed by the help of a plane table, which is a rectangular board, ſupported by a fulcrum, ſo as to turn every way, by means of a ball and ſocket. It has a moveable frame ſurrounding it, for keeping the paper laid on it tight to it: one ſide of the frame is divided into equal parts, all around it, and the other ſide is divided into 360 degrees, by lines directed to a point in the middle of the board. The board has alſo a compaſs, and a large index, with either teleſcopic or open ſights.

PROB. XVIII.

Pl. II. Fig. 29.

TO delineate a field, by the help of a plane table, from one ſtation ſo taken, that all the angles may be ſeen from it, and their diſtances meaſured.

Let the field be ABCDE. Let the table be erected at any convenient place, and fix the paper on it, in which let ſome point F be taken near the middle, to repreſent the ſtation. Then applying the index to this point, direct it to one of the angles, as A, and from F draw a faint line along the edge of the index. Then meaſure the diſtance of that angle from the ſtation, and ſet it off on the faint line from F, by the line of equal parts, and at its end make a mark to repreſent the angle A of the field. The ſame is to be done with the reſt of the angles, and with every remarkable point neceſſary to be put on the plan. And if the marks repreſenting the angles be joined, the figure on the table will be ſimilar to the field, by the 5th Prop. B. 6. of Euclid.

Cor. It is evident, that the ſame thing may be done by the theodolite; for having taken all the angles at the point F, and meaſured the lines from the ſtation to the angles of the field, draw triangles ſimilar to theſe on the paper, and the figure thus made will repreſent the field. But the angles muſt be examined, for they ſhould be together 360°, by the 32d. B. 1. of Eucl.

Note, The ſurveyer ſhould have a field-book ruled in columns, in which he is to place diſtinctly all the angles and ſides, and the offsets, croſs-hedges, houſes, ponds, ſtiles, brooks, &c.

PROB.

PROB. XIX.

Part I. Fig. 30. Pl. II.

To lay down a field, by means of two ſtations, from each of which all the angles can be ſeen.

Place the inſtrument at the ſtation F, and having choſen a point on the paper upon the table, to repreſent it, apply the index to it, and direct it ſucceſſively to all the angles, and draw lines along its edge: Then direct the index to the other ſtation G, and draw a line, on which ſet off the diſtance of the ſtations, from a line of equal parts. Next remove the inſtrument to the ſtation G, and applying the index to the line joining F and G, turn the inſtrument until the firſt ſtation is ſeen through the ſights; there fix the inſtrument, and applying the index to G, direct it to all the angles, and draw lines as before: and the interſection of the lines drawn to the ſame angle will repreſent that angle: and lines joining theſe interſections will form a figure like the field.

Cor. It is evident, that the ſame may be eaſily done by the theodolite, by taking the angles which the line between the ſtations makes with the neareſt angles of the field, and the angles at the ſtations ſubtended by the ſides of the field; and this is the common way, eſpecially in laying down large fields, farms, or eſtates.

Note, Great care ſhould be taken in chuſing the ſtations; their diſtance ſhould not be too ſmall, nor the angles ſubtended by it, at the corners of the field, very acute, or very obtuſe.

PROB. XX.

Pl. II. Fig. 31.

To lay down a field by going round it.

Let ABCEDHG be the field. In going round it, meaſure its ſides, and take its angles with the theodolite. Then let a figure, ſimilar to it, be drawn upon paper. If there be a mountain or hollow in any ſide, its horizontal diſtance is to be taken, and the place of the mountain is to be ſhaded. If any place within the field, as F, is to be laid down, take the angles BAF and ABF, or rather meaſure AF and BF, and lay down the triangle AFB.

Note 1. Be careful to diſtinguiſh exterior angles, as E and G, that they may be made ſuch on the plan. And to know if the angles be right, ſubtract each of the exterior angles from 360 degrees, and add the remainders to the other angles and

360 degrees; the ſum divided by 90 degrees, gives twice the number of ſides [a].

[a] Cor. 32. 1. E.

PROB. XXI.

Pl. II. Fig. 32.

To lay down a plane field with the chain.

Meaſure round it, and likewiſe meaſure lines acroſs, from corner to corner, ſo as to divide it into triangles. Or after meaſuring round it, meaſure the diſtance of ſome ſtation C, from its ſeveral angles, then lay down one of the triangles, as ABC, and upon BC deſcribe the next triangle BCD, and ſo on; then, becauſe each triangle on the paper is ſimilar to that in the field, the whole is like the whole.

SCHOLIUM.

If the field be ſmall, and all its angles can be ſeen from one ſtation, it is to be meaſured by Prob. 18. If it be large, and great exactneſs is not required, it may be meaſured by Prob. 19. of this. But in fields that are irregular, or mountainous, we are to proceed as in Prob. 21. of this, and the 20th Problem may be ſometimes of uſe, for we can truſt more to the meaſuring of ſides, than to the obſerving of angles.

It is neceſſary to take the bearing of ſome line in the field, by the compaſs, in order to give it its true ſituation on the paper, and to connect it properly with the adjacent fields.

After the figure is drawn upon paper, with the ſcale and compaſſes, it is to be transferred to clean paper, by laying the draught upon the clean paper, and marking the angles, and remarkable points, with the point of a ſmall needle, by means of which points it can be drawn anew; and then let the whole be illuminated, by colours proper to each part, and make a compaſs in ſome convenient part of it, and a ſcale in the margin.

PART

PART II.

OF SUPERFICIES.

PART II.

The ſmalleſt ſuperficial meaſure with us is the ſquare, each ſide of which is an inch in length; 144 ſquare inches make a ſquare foot; 9 ſquare feet make a ſquare yard; $30\frac{1}{4}$ ſquare yards a pole; 40 poles a rood; and 4 roods an acre.

In Scotland, they meaſure land by the ſquare ell, which contains 1369 of the Scots ſquare inches, or 1383.84 Engliſh ſquare inches: 36 ſquare ells make a fall; 40 falls a rood, and 4 roods an acre. But ſurveyers in Scotland often make the Scots ell to conſiſt of 37 Engliſh inches; and thus make the acre too little, by 593.6 Engliſh ſquare feet; for the Scots acre is to the Engliſh, as 10,000 to 78,694. 6 acres, arable land, make a huſband-land; 13 acres an oxgate; 4 oxgate a pound-land, and 8 oxgate a forty-ſhilling land.

In meaſuring land by the chain, the dimenſions are expreſſed in links, and the contents in ſquare links. Now, the chain being 100 links, a ſquare chain is 10,000 ſquare links, and 10 ſquare chains, or 100,000 ſquare links make an acre; therefore, to reduce ſquare links to acres, cut off 5 decimal places, and the reſt are acres: and the decimals may be reduced to roods, by multiplying by 4, and cutting off the ſame number of decimals from the product; and ſo on.

The French arpent contains 32,400 ſquare feet of Paris, and is equal to 3.37 Engliſh roods.

PROB. I.

To find the area of a rectangle ABCD.

Pl. III. Fig. 1.

Let the ſide AB, for example, be 5 feet, and let it be divided into 5 equal parts, one of which is AG; and let AE be equal to AG, and complete the parallelograms GE, EB. Then the rectangle EG : EB : : AG : AB [a], that is, as 1 to 5; therefore EB is 5 times EG. Let AD be three feet; then, becauſe BE : BD : : AE : AD, that is, as 1 to 3, BD is 3 times BE, and BE is 5 times EG; therefore BD is 15 times EG. Therefore the area is found, by multiplying the length by the breadth.

a 1. 6. E.

Cor. If the parallelogram be not a rectangle, it is equal to a rectangle, on the ſame baſe, and between the ſame parallels [b]; therefore,

b 35. 1. E.

therefore, multiply its baſe AB into its perpendicular height AD, (not into its ſide AH), and the product is the area.

PROB. II.

Pl. III. Fig. 2. TO find the area of a triangle.

a 41. 1. E. A triangle being the half [a] of a rectangle upon the ſame baſe with it, and between the ſame parallels: Therefore, from the angle A, draw AD, perpendicular to the oppoſite ſide BC, and meaſure BC and AD. Then multiply one of them by the half of the other, or multiply the two together, and take the half of the product. Thus, if BC be 9 feet long, and AD 6 feet, the area of the triangle will be 27 ſquare feet.

When the three ſides are given, the perpendicular may be found, by protracting the triangle, and meaſuring the perpendicular upon the ſame ſcale, or it may be found by the 12th and 13th 2. Eucl. or by Trigonometry.

PROB. III.

Pl. III. Fig. 3. THE ſides of a triangle being given to find the area, without finding the perpendicular.

Let ABC be the triangle, and make AD equal to AC, and
a 10. 1. E. AE to AB, and join DC, BE, and biſect [a] DC in F, and join
b 31. 1. E. AF, meeting BE in G, and draw [b] FL parallel to BC, meeting
h 34. 1. E. AB, BE in K, L; and LB is equal [h] to FC or DF: and becauſe
c 2. 6. E. AD is equal to AC, and DB to CE, BE is parallel to DC [c]:
d D. 5. E. and becauſe DC is double of DF, DB is double of DK [d], and
BC or LF double of FK, and BE of BG: and becauſe BA, AG
are equal to EA, AG, and the baſe BG is equal to GE; the
e 8. 1. E. angle BAG is equal [e] to GAE, and the angle BGA to [f]
f 4. 1. E. EGA; therefore they are right angles, and the circle deſcribed from the centre K, at the diſtance KF, ſhall paſs through
g 2. Cor. G [g]; let this be the circle FGL, meeting AB in H, M; there-
A. 3. E. fore HM is equal to FL, or BC the baſe: and becauſe AD is equal
to AC, and DK to KB, AK is half the ſum of AB, AC; and
KH is the half of BC; therefore AH is half the perimeter, and
AM is its exceſs above HM or BC: alſo, becauſe HK is equal to
KM, and KD to KB, BM is equal to HD the exceſs of AH
above AD or AC, and HB is its exceſs above AB: But the

rectangle HB, BM is equal [k] to the rectangle GB, BL, or GB, DF; and the rectangle HA, AM is equal to the rectangle GA, AF [l]: and becauſe BG is equal to GE, the triangle ABE is equal [m] to the rectangle BG, GA, and the triangle CBE to the rectangle BG, GF; therefore the triangle ABC is equal to the rectangle BG, AF: and becauſe BG is to GA, as DF to FA [n], the rectangle BG, AF is equal [o] to the rectangle AG, DF: But as the rectangle GA, AF to BG, AF, ſo is AG to GB [p], and ſo is AG, DF or BG, AF to BG, DF [p]: and the rectangle GA, AF is equal to HA, AM; and BG, AF to the triangle ABC; and BG, DF to HB, BM; that is, the triangle is a mean proportional between the rectangle contained by half the perimeter, and its exceſs above the baſe, and the rectangle contained by the exceſſes of half the perimeter above the other two ſides. Wherefore, add the three ſides together, and from half the ſum, ſubtract the ſides ſeparately; then multiply the half ſum, and the three differences into one another, and the ſquare-root of the product is the area of the triangle. For example, let the ſides be 10, 17, and 21; the half of their ſum is 24, and its exceſſes above the ſides are 14, 7, and 3; and 24 multiplied by 3, and the product by 7, and this laſt product by 14, gives 7056; the ſquare-root of which 84 is the area.

[k] 35. 3. E.
[l] Cor. 36. 3. E.
[m] 41. 1. E.
[n] 4. 6. E.
[o] 16. 6. E.
[p] 1. 6. E.

PROP. IV.

To find the area of any rectilineal figure.

Pl. III. Fig. 4.

Divide the figure into triangles, by joining its angles; and find the area of each of the triangles, by the 2d or 3d Prob. and the ſum of theſe areas will be the area of the figure.

Fig. 5.

If the figure be quadrilateral, and have two ſides parallel to one another, multiply half the ſum of the parallel ſides by the perpendicular drawn from one of them to the other, and the product is the area. For example, let the parallel ſides be 14 and 12, and the perpendicular 15, then 15 multiplied by 13, which is half the ſum of 14 and 12, gives 195 for the area.

Note, If a field is to be meaſured, let it firſt be laid down on paper, and divided into triangles; then any line, or the diſtance of two points of it, may be meaſured, by applying it to the ſcale from which the figure is drawn. But the ſcale gives the horizontal line, or the neareſt diſtance between two points, and not their diſtance in the field, which is commonly uneven; if this be wanted, it muſt be meaſured in the field. But the horizontal plane, on which an uneven or hilly field ſtands, is that upon which buildings are erected, and is that alone which is occupied by trees or corn.

PROB. V.

Pl. III. Fig. 6.

TO find the area of a regular polygon ABEFGH.

Let C be the centre of the circle deſcribed about the polygon, and it is evident, from 12. 4. E, that all the perpendiculars drawn from C to the ſides are equal; therefore, if lines be drawn from C to the angles, the polygon will be reſolved into as many equal triangles as it has ſides; and ſince the product of the perpendicular into the half of one of the ſides is the area of one of the triangles, the product of the perpendicular into half the ſum of the ſides, will be the ſum of the triangles; that is, the polygon. Wherefore, multiply half the perimeter, by the perpendicular from the centre to one of the ſides; the product is the area.

COR. Since 90 degrees, multiplied by the number of ſides, wanting two, is half the ſum of the angles of the figure[a], if this product be divided by the number of ſides, the quotient is one of the angles at the baſe of the triangles, as CAB: and, by Trigonometry, R : tan. CAB : : $\frac{1}{2}$ AB : perpendicular CD. By this corollary, and the problem, the following table was conſtructed, when the ſide is unit: and by it the area may be found, by multiplying the tabular number by the ſquare of the ſide.

[a] Cor. 32. 1. E.

Names.	Numb. of ſides.	Angles at centre.	Angles CAB.	Perpendiculars.	Areas.
Equilat. triang.	3	120°	30°	0.2886752	0.4330127
Square.	4	90	45	0.5000000	1.0000000
Pentagon.	5	72	54	0.6881910	1.7204774
Hexagon.	6	60	60	0.8660254	2.5980762
Heptagon.	7	$51\frac{3}{7}$	$64\frac{2}{7}$	1.0382601	3.6339124
Octagon.	8	45	$67\frac{1}{2}$	1.2071069	4.8284271
Nonagon.	9	40	70	1.3737385	6.1818242
Decagon.	10	36	72	1.5388418	7.6942088
Undecagon.	11	$32\frac{8}{11}$	$73\frac{7}{11}$	1.7028439	9.3656399
Dodecagon.	12	30	75	1.8660252	11.1961524

Let the ſide of a regular hexagon be 14; the ſquare of the ſide is 196, which, multiplied by 2.5980762, gives 509.2229352 for the area of the hexagon.

PROB. VI.

TO find the area of a circle.

Let ADCE be the circle, of which the radius AB is 7, and the circumference 44. Multiply 22, the half of the circumference, by 7, and the product 154 is the area of the circle[a].

Pl. III. Fig. 7.

a 2. 12. E.

Cor. 1. Becauſe the radius is to the half of the circumference, as 1 to 3.1416, as was ſhewn at the end of Plane Trigonometry; it is manifeſt, that the ſquare of the radius is to the area of the circle, as 1 to 3.1416[b], and therefore the area is equal[c] to the ſquare of the radius, multiplied by 3.1416, or to the ſquare of the diameter, multiplied by .7854 the fourth part of 3.1416, becauſe the ſquare of the diameter is four times[d] the ſquare of the radius. Thus, if the diameter be 14, its ſquare 196, multiplied by .7854, gives 153.9384 for the area of the circle.

b 1. 6. E.

c 16. 6. E.

d 3. Cor. 4. 2. E.

Note, When the diameter is 1, the circumference has been found to be 3.14159,26535,89793,23846,26433,83279,50288,41971,69399,37510,58209,74944,59230,71864,06286,20899,86280,34825,34211,70679+&c. And therefore, if very great accuracy is required, uſe this number inſtead of 3.1416.

Cor. 2. Hence, it appears, that the area of any ſector of a circle, as ABCD, is found, by multiplying the half of the arch, by the radius; and that the area of the ſegment ADC is got by ſubtracting the area of the triangle ABC, from the area of the ſector.

PROB. VII.

TO find the area of an ellipſe.

Pl. III. Fig. 8.

It is proved, by the writers on Conic Sections, that an ellipſe is a mean proportional between the circles, which have its axes for their diameters. Therefore, multiply the product of the diameters by .7854, and the product is the area of the ellipſe. For example, let the greater axis BD be 10, and the leſſer AC 7, their product 70, multiplied by .7854, gives 54.978, which is the area of the ellipſe.

Cor. Hence, as one of the axes is to the other, ſo is the circle upon that axis to the ellipſe; and ſo is any ſegment of the circle cut off by a perpendicular to that axis, to the ſegment of the ellipſe cut off by the ſame line.

PROB.

PROB. VIII.

Part II.

To find the area of an offſet.

Pl. III. Fig. 9.

Offsets are parts of a field which lie between a crooked hedge and the ſtraight line which is meaſured for the ſide of the field; and theſe ſpaces are meaſured, by taking the perpendicular diſtances of the hedge from the meaſured line, at a great number of places, ſo that the hedge between theſe places may be nearly a ſtraight line; and then the ſpaces between the perpendiculars are found by the rule for finding the area of a quadrilateral, given in Prob. 4. of this Part. Or if the meaſured ſide be divided into equal parts by the perpendiculars, and meet the hedge at both its ends, add the perpendiculars together, and divide the ſum by the number of parts into which they divide the baſe, and the quotient is a mean perpendicular, which, multiplied by the baſe, gives the area. If there be perpendiculars at the ends of the baſe, half their ſum ſhould be added to the other perpendiculars before diviſion. But the common rule is to add all the perpendiculars together, and divide by the number of them, for a mean perpendicular to be multiplied by the baſe. This rule, however, gives the area too great, when there are not perpendiculars at the ends of the baſe, and too little when there are. Let the baſe be 44, and the perpendiculars at its ends 8 and 16, and the other perpendiculars 14, 16, and 17, the ſum of theſe three is 47, which, added to 12 half the ſum of 16 and 8, gives 59, which, divided by 4 the number of parts of the baſe, gives $14\frac{3}{4}$ for the mean perpendicular; and this, multiplied by the baſe 44, gives the area 649.

But the ſum of all the perpendiculars 71, divided 5 their number, gives $14\frac{1}{5}$ for the mean perpendicular, and this, multiplied by 44, makes the area 633.6 only.

PROB. IX.

To find the ſuperficies of any priſm.

The baſes, if they are regular figures, are meaſured by Prob. 5. of this Part; or if they are irregular, they are meaſured by Prob. 4. and the ſides being parallelograms, are meaſured by Prob. 1. of this Part: and the ſum of theſe is the ſuperficies of the priſm.

PROB.

PROB. X.

TO meaſure the ſuperficies of a pyramid.

Part II.

The baſe is meaſured by Prob. 5. if it be regular, or by Prob. 4. if it be irregular; and the ſides being triangles, are meaſured by Prob. 2. or 3. of this Part.

PROB. XI.

TO meaſure the ſuperficies of any regular body.

Regular bodies are thoſe bounded by equilateral and equiangular figures. The ſuperficies of the tetrahedron conſiſts of four equal and equilateral triangles; the hexahedron, or cube, of ſix equal ſquares; the octahedron, of eight equal equilateral triangles; the dodecahedron, of twelve equal regular pentagons; and the ſuperficies of the icoſahedron, of twenty equal equilateral triangles. Therefore it will be eaſy to meaſure them from what has been ſhown. And in the ſame manner may the ſuperficies of any ſolid contained by planes, be meaſured.

PROB. XII.

TO meaſure the ſuperficies of a cylinder ABCD.

Pl. III. Fig. 10.

Let AE be equal to the circumference of the baſe, and complete the rectangle DE; it is equal to the ſuperficies of the cylinder. Let AF be any line leſs than AE, and in the baſe, let a polygon be inſcribed, greater than the rectangle contained by FA, and the radius AO, and at its angles, let perpendiculars to the baſe be erected, which will conſtitute a priſm within the cylinder, of which the ſuperficies is leſs than that of the cylinder: but by the 9th of this part, the ſuperficies of the priſm is equal to the rectangle contained by DA, and the perimeter of the polygon, which is greater than AF, becauſe the rectangle contained by AO, and that perimeter, is greater [a] than the polygon, and therefore greater than the rectangle FA, AO; much more, therefore, is the ſuperficies of the cylinder greater than the rectangle DA, AF, that is, than any rectangle leſs than AE. And in the ſame manner, it may be proved, that it is leſs than any rectangle greater than AE; wherefore, it is equal to AE: Therefore,

[a] 5. of this.

Therefore, multiply the circumference of the baſe by the altitude, and the product is the ſuperficies of the cylinder, excluding the baſes, which, being circles, are meaſured by Prob. 6.

Note, It is evident, that if the ſuperficies of the cylinder be ſpread out, it will coincide with DE.

PROB. XIII.

Pl. III. Fig. 11.

To meaſure the ſuperficies of a cone ABC.

Let BE be at right angles to AB, and equal to the circumference of the baſe, and join AE; and the triangle ABE is equal to the ſuperficies of the cone: let BF be any line leſs than BE, and join AF, and in the baſe, let a polygon be inſcribed greater than the rectangle FB, BO, and draw lines from A, to all its angles, which will conſtitute a pyramid within the cone: let HK be a ſide of the polygon, biſected by BC in G, and join AG; and as OG to GA, ſo make [a] BG to GL; therefore GL is greater [b] than GB, and AL greater than AB; alſo OG is to GA, as [c] OB to AL; and therefore the rectangle contained by OG, and the perimeter of the polygon, is to the rectangle contained by AG, and the ſame perimeter, as [d] the rectangle OB, BF to the rectangle AL, BF; that is, the polygon [e] is to twice the ſuperficies [f] of the pyramid, as OB, BF to AL, BF: but the polygon is greater than the rectangle OB, BF; therefore twice the ſuperficies of the pyramid is greater [g] than the rectangle AL, BF, and therefore much greater than the rectangle AB, BF: Wherefore, the ſuperficies of the pyramid is greater than the triangle ABF, and much more is the ſuperficies of the cone greater than any triangle ABF, that is, leſs than ABE. In the ſame manner, it may be proved, that it is leſs than any triangle greater than ABE. Therefore, the ſuperficies of the cone is equal to the triangle ABE; and therefore its area is got by multiplying the half of the circumference of the baſe, by the ſlant ſide AB. This is alſo proved, by ſuppoſing the ſuperficies to be ſpread out on a plane, for it will then be a ſector, of which AB is the radius, and the circumference of the baſe is the arch.

[a] 12. 6. E.
[b] A. 5. E.
[c] 12. 5. E.
[d] 1. 6. E.
[e] 5. of this.
[f] 10. of this.
[g] 14. 5. E.

Cor. It is evident, that if the cone be cut by a plane at M parallel to the baſe, and MN is parallel to BE, the ſuperficies of the fruſtum is equal to the quadrilateral BENM, of which BE and NM are parallel ſides, and BM their diſtance.

PROB. XIV.

To meaſure the ſuperficies of a ſphere.

PART II.

Pl. III. Fig. 12.

Let ABC be a ſemicircle, of which the diameter is AC and the centre D; and let GHKL be any regular polygon deſcribed about it; and draw [a] MO, HQ, NP, KR, perpendiculars to AC, and HS parallel to it [b]; and join DM, DN, and complete the parallelogram CE. And becauſe the triangles GMD, GQH, right angled at M, Q, are equiangular [c], GM : MD : : QG : QH [d]; and, alternately [e], GM : GQ : : MD : QH, that is, as the circumference of which MD is the radius to the circumference of which QH is the radius [f]; therefore the rectangle contained by GM and the latter circumference is equal to the rectangle contained by GQ and the former [g]: But the ſuperficies of the cone, deſcribed by the revolution of the triangle GQH about GQ, is equal [h] to the rectangle contained by GM, and the circumference of which QH is radius; therefore, alſo, it is equal to the rectangle contained by GQ and the circumference ABC of which DM is radius. Again, becauſe the angles SKH, KHS are equal to the right angle HND [c], and SKH is equal [k] to HNP, the angle KHS is equal PND; therefore the right angled triangles KHS, DPN are equiangular [c], and DN is to NP, as [d] KH to HS, or QR; and therefore, it may be proved, as before, that the ſuperficies deſcribed by the revolution of HKRQ about QR, which is equal to the rectangle contained by HK, and the circumference of NP, half the ſum of QH and RK, is alſo equal to the rectangle contained by QR and the circumference of the inſcribed circle ABC; and ſo on: Wherefore, the whole ſuperficies deſcribed by the revolution of the polygon, is equal to the rectangle contained by the axis GL, and the circumference of the inſcribed circle ABC: but the circle ABC is equal to the baſe of the cylinder, deſcribed by the revolution of the rectangle EC, and the axis GL is greater than the height AC; therefore the ſuperficies deſcribed by the polygon GKL is greater than the ſuperficies of the cylinder deſcribed by CE. Let, in the ſame manner, any polygon be inſcribed in the ſemicircle ABC, and it may be proved, as before, that the ſuperficies deſcribed by its revolution is equal to the rectangle contained by its axis AC, and the circumference of the circle inſcribed in it: but this circle is leſs than the baſe of the cylinder, and AC is the height of the cylinder; therefore the ſuperficies deſcribed by the polygon is leſs than that of the cylinder. Wherefore, the ſuperficies of the cylinder is leſs than any ſuper-

[a] 12. 1. E.
[b] 31. 1. E.
[c] 32. 1. E.
[d] 4. 6. E.
[e] 16. 5. E.
[f] Cor. 3. 12. E.
[g] 16. 6. E.
[h] 13. of this.
[c] 32. 1. E.
[k] 29. 1. E.
[d] 4. 6. E.

ficies greater than that of the ſphere, and greater than any leſs than that of the ſphere; it is therefore equal to the ſuperficies of the ſphere. And likewiſe any ſegment of the cylindric ſuperficies is equal to the ſuperficies of the correſponding ſegment of the ſphere, cut off by the ſame plane.

Cor. 1. The ſurface of the ſphere is four times a great circle of it, for a great circle is equal to the rectangle contained by half the circumference and half the diameter.

Cor. 2. Becauſe the rectangle CA, AP is equal [m] to the ſquare of the chord AN, if each of them be multiplied by 3.1416, the circle of which AN is the radius is equal [n] to the rectangle contained by AP, and the circumference of the circle ABC; that is, it is equal to the ſuperficies of the ſegment made by the revolution of AMN about AP.

[m] C. 3. E.

[n] 4. of this.

PART III.

OF SOLID FIGURES.

As an inch is the ſmalleſt meaſure in length, and the ſquare upon it the ſmalleſt ſuperficial meaſure, ſo the cube deſcribed from an inch is the ſmalleſt ſolid meaſure; and 1728 cubical inches make a ſolid foot.

The Engliſh ale-gallon contains 282 cubical inches; a pint is the eighth part of a gallon, and contains $35\frac{1}{4}$ cubical inches; in the country, 34 gallons make a barrel; but in London, 32 gallons make a barrel of ale, and 36 gallons a barrel of beer; and a firkin is the fourth part of a barrel.

The Engliſh wine-gallon contains 231 cubical inches, and therefore the pint contains $28\frac{7}{8}$ cubical inches; there are 63 gallons in a hogſhead, 84 gallons in a puncheon, 2 hoſheads in a pipe, and 4 hoſheads in a ton; and a tierce is the half of a puncheon.

The Scots pint contains $103\frac{4}{10}$ cubical inches, but is ſuppoſed to contain 105 ſuch inches; 2 pints make a quart, and 8 pints a gallon; and the pint is ſubdivided into 2 chopins, or 4 mutchkins, and the mutchkin into 4 gills.

In dry meaſure, the Wincheſter gallon contains $272\frac{1}{4}$ cubical inches; and 8 gallons make a buſhel, which ſhould therefore contain 2178 cubical inches: but in levying the malt-tax, the buſhel is appointed to contain 2150 cubical inches; and the gallon anſwering to it ſhould be $268\frac{4}{5}$ cubical inches; 2 gallons make a peck, and 32 pecks, or 8 buſhels, a quarter.

The Scots wheat-firlot contains $21\frac{1}{4}$ pints, or 2199 cubical inches, and the barley-firlot 31 pints, or 3208 cubical inches; 4 firlots make a boll, and 16 bolls a chalder; a peck is the fourth part of a firlot, and a lippie the fourth part of a peck.

A Paris pint is nearly equal to 2 Engliſh pints, and contains 48 cubical Paris inches.

PROB. I.

TO find the content of a priſm.

If it be a rectangular parallelopiped, of which the altitude is one inch, it is evident, that there will be as many cubical inches in its content, as there are ſquare inches in its baſe; and if its altitude be any number of inches, the content will be as many times the number of cubical inches; therefore find the area of the baſe, and multiply it by the height, and the product is the ſolid content: and every priſm is equal [a] to a rectangular parallelopiped, of the ſame baſe and altitude with it. Therefore, if the baſe found by the 4th or 5th Problem of the 2d Part be 96 inches, and the height 11 inches, the content is 1056 ſolid inches.

a 2. Cor. 32. 11. E.

PROB. II.

TO find the ſolid content of a pyramid.

Find the area of the baſe by the 4th or 5th Problem of the 2d Part, and multiply it by $\frac{1}{3}$ of the height, the product is the content, by 4th Prop. 12th Book E.

COR. If the ſolid content of a fruſtum of a pyramid be required, find the content of the entire pyramid, and of the part that is wanting, and their difference is the content of the fruſtum: or find the areas of the baſes, and multiply them by correſponding ſides of theſe baſes; then, as the difference of theſe ſides to the height of the fruſtum, ſo [a] is the difference of the products to the content.

a 4. 6. and 19. 5. E

PROB. III.

TO find the ſolid content of a cylinder.

a 6. 11. E. and 1. of this.

Find the area of the baſe, by Prob. 6. or 7. of the 2d Part, and multiply it by the height, the product is the ſolid content [a]: Thus, if the diameter of the baſe be 14, and the height 12 inches, the area of the baſe is 153.9384, which, multiplied by 12, gives 1844.2608 ſolid inches for the content.

Pl. III. Fig. 13.

Cor. If the bung-diameter EF of a caſk ABCD be not much greater than the head-diameter AB; find the areas of the circles at the head and bung, and take half the ſum for a mean baſe, which, multiplied by the length, gives the content; only the dimenſions are to be taken within the ſtaves.

PROB. IV.

To find the ſolid content of a cone.

a 1. Cor. 7. 12. E.

Find the area of the baſe, by Prop. 6. 2d Part, and multiply it by $\frac{1}{3}$ of the height, to get the ſolid content [a]. For example, if the diameter of the baſe be 8 inches, and the height 9 inches, the area of the baſe is 50.2656, which, multiplied by 9, gives 452.39 ſolid inches for the content.

Pl. III. Fig. 14. b 31. 1. E. c 4. 6. E.

Cor. 1. If the content of the fruſtum of a cone be required, let it be ABCD; draw [b] AG, parallel to DE; then GC : CD : : AC : CE : : AH : EF [c], the altitude of the complete cone; and if the content of the complete cone ECD, and of the part EAB, be found, their difference is ABCD. Which may be done, by ſubtracting the product of the diameters CD and AB from the ſquare of their ſum, the remainder, multiplied by .7854, and then by $\frac{1}{3}$ of the height AH, gives the content of the fruſtum.

Pl. III. Fig. 15.

Cor. 2. Caſks, of which the ſtaves are very much bended towards the middle, and ſtraight towards the ends, as ABEF, may be taken for two fruſtums of a cone.

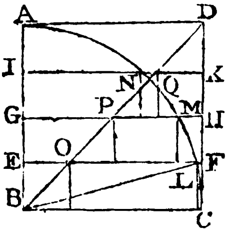
PROB. V.

To find the ſolid content of a ſphere.

a 9. 12. E.

Becauſe the ſphere is $\frac{2}{3}$ of its circumſcribed cylinder [a], find the area of a great circle of the ſphere, and multiply it by $\frac{2}{3}$ of the axis, the product is the content: or, which is the ſame thing, multiply the cube of the axis by $\frac{2}{3}$ of .7854, that is, .5236. For example, let the axis be 9 inches, the cube of it is 729, which, multiplied by .5236, gives 381.7044 for the ſolid content.

Cor. 1. It may be proved, as was done in the 9th Propoſition of the 12th Book, that the ſolid deſcribed by the revolution of the part BGMC of the circle, together with the cone deſcribed by the triangle BGP, is equal to the cylinder deſcribed by the rectangle BH; and therefore the content of the zone deſcribed by BGMC is got by ſubtracting $\frac{1}{3}$ of the ſquare of GP or GB the height, from the ſquare of BC [b], and multiplying the remainder by 3.1416, and then by the height BG.



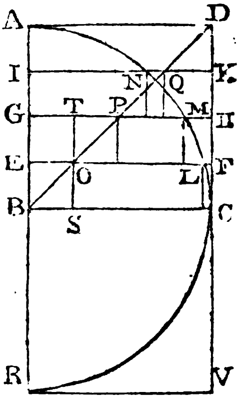
b 3. 12. E.

Cor. 2. Hence the content of a ſegment of the ſphere may be found, by ſubtracting the remaining zone from the hemiſphere.

PROB. VI.

To find the ſolid content of a ſpheroid.

A ſpheroid is a ſolid deſcribed by the revolution of a ſemiellipſe about one of its axis.



If ACR be a ſemiellipſe, and the preparation be made as in the 9th Propoſition of the 12th Book, it may be proved, that the ſpheroid deſcribed by the ſemiellipſe ACR, about the axis AR, is $\frac{2}{3}$ of the circumſcribing cylinder: For, by the nature of the ellipſe, the ſquare of EL is to the rectangle AE, ER, as the ſquare of DA is to that of AB, that is, as [a] the ſquare of EO to that of EB; therefore the ſquares of EL, EO are to the rectangle AE, ER, and the ſquare of EB, as the ſquare of DA to that of AB [b]; but the rectangle AE, ER, with the ſquare of EB, is equal [c] to the ſquare of AB; therefore the ſquares of LE, EO are equal [d] to that of AD or EF; and the cylinders deſcribed by the revolution of the rectangles BL, BO, are therefore equal [e] to the cylinder deſcribed by BF. From which it may be proved, that the cone deſcribed by BAD, together with the hemiſpheroid deſcribed by AMC, is equal to the cylinder deſcribed by AC, as was done in the 9th Prop. of the 12th Book. And the cone is a third part of the cylinder [f]; therefore the hemiſpheroid is $\frac{2}{3}$ of it. Therefore, find the area of the circle deſcribed by the revolving axis, and multiply it by $\frac{2}{3}$ of the fixed axis.

a 4. 6. E.
b 12. 5. E.
c 5. 2. E.
d 14. 5. E
e 1. Cor. 6. 12. E.
f 1. Cor. 7. 12. E.

Cor. 1.

Cor. 1. In the ſame manner, it may be proved, that the part of the ſpheroid deſcribed by the revolution of BGMC, together with the cone deſcribed by BGP, is equal to the cylinder deſcribed by BGHC. Now, the circle deſcribed by GP is the exceſs of the circle deſcribed by GH or BC, above that deſcribed by GM; therefore, from three times the area of the greater baſe, ſubtract the exceſs of this area above the area of the leſs baſe; or, which is the ſame thing, to twice the area of the greater baſe add the area of the leſs baſe, and the ſum is the baſe of a cone, equal [f] in height and content to the zone of the ſpheroid.

[f] 1. Cor. 7. 12. E.

Pl. III. Fig. 16. Cor. 2. When the ſtaves of a caſk are much bended, it is ſuppoſed to be the middle zone of a ſpheroid: Let the length of ſuch a caſk be 40 inches, and its bung-diameter 32 inches, and its head diameter 26 inches; to 2048, twice the ſquare of 32, add 676, the ſquare of 26, and multiply the ſum 2724 by 40, the length, and then by .2618, $\frac{1}{3}$ of .7854, and the product 28525.728 is the ſolid content in inches, which, divided by 282, gives 101 gallons 1 pint of ale; or if it be divided by 231, the quotient is 123 gallons 1 pint of wine.

PROB. VII.

To find the ſolid content of any regular body.

The ſolidity of the tetrahedron, which is a pyramid, is found by the 2d Problem of this Part. The hexahedron, or cube, is found, Prob. 1. of this. The octahedron, being two pyramids of equal heights, upon the ſame ſquare for a baſe, is meaſured by Prob. 2. The dodecahedron conſiſts of 12 equal pyramids, upon regular pentagons for their baſes, and may be meaſured by Prob. 2. of this. The icoſahedron conſiſts of twenty equal pyramids, upon triangular baſes, and may be meaſured by the ſame problem. The baſes and heights of theſe pyramids may be either meaſured, or elſe found by Trigonometry.

PROB. VIII.

To find the ſolidity of any body, however irregular.

Let the body be immerſed in water, in a veſſel of the figure of a priſm, and take notice how much the water is raiſed by the immerſion of the body: for it is plain, that the ſpace which the water fills, after immerſing the body, exceeds the ſpace it occupied before, by a ſpace equal to the ſolid content of the body; and this exceſs may be found, by Prob. 1. by multiplying the area of the mouth of the veſſel, by the difference between the elevations of the water before and after immerſion. In the ſame manner, the ſolidity of a part of a body may be found, by immerſing that part only.

PROB. IX.

To find how much is contained in a caſk, that is in part empty, of which the axis is parallel to the horizon.

Pl. III. Fig. 17. & 18.

Let AGBH be the circle made by cutting the caſk at the bung, by a plane perpendicular to the axis, AB the bung-diameter, and GBH the ſegment of it, filled with liquor, of which the depth EB may be found by the guaging rod; and the diameter AB, being alſo known, the area of the ſegment GBH may be found, by the 2d Cor. to Prob. 6. Part 2. or it may be found, from a table of ſegments, ſuch as that at the end of guaging.

Find the mean baſe of the caſk, which, if the caſk be not of the kind mentioned in Prob. 3. of this Part, may always be found, by dividing the content of the caſk by its length; let this be the circle CKDL, and let the ſegment of it, ſimilar to GBH, be KDL; this ſegment may be found, for the circle AGBH is to CKDL, as the ſegment GBH to the ſegment KDL; and this ſegment, multiplied by the length of the caſk, gives the quantity of liquor in it.

OF GUAGING.

To guage a veſſel, is to find its content in the meaſures of any country; and this may be done, by means of the preceding problems, by firſt finding the ſolid content in inches, and then dividing it by the number of inches in the meaſure required: but the work may be ſhortened, by finding multipliers, which will give the content in that meaſure, inſtead of inches. For example, if the veſſel be contained by planes, inſtead of dividing by 231, multiply by the decimal .004329, and the product will be the content in wine-gallons; and inſtead of dividing by 282, multiply by .003546, to get the content expreſſed in gallons of ale: But thoſe multipliers are ſeldom uſed, becauſe they do not abridge the operation; they are found by annexing cyphers to unity, and dividing by the number of inches in the wine or ale gallon. The multipliers for round veſſels are got, by dividing .7853982 by the number of inches in the ſeveral meaſures in which it is required that the contents ſhould be expreſſed. For example, .7853982, divided by 282 the number of inches in an ale-gallon, gives .0027851, the multiplier for ale-gallons; therefore, inſtead of multiplying by .7854, in finding the content, multiply by .0027851, and the content will be found in ale-gallons. In the ſame manner, .0034 is found to be the multiplier for wine-gallons: and ſince theſe multipliers are to one another, nearly as 9 to 11, it follows, that 9 ale-gallons are nearly equal to 11 wine-gallons. And as there are 2150 ſolid inches in a malt-buſhel, the multiplier in this caſe will be .0003653: and if the Scots pint be 103.4 inches, the multiplier for it will be .0076; and for the Scots wheat-firlot of 21¼ pints, or 2197 inches, the multiplier is .000358; or for the barley-firlot, of 31 pints, it is .000245; ſo that 107 wheat-firlots are nearly equal to 110 malt-buſhels; and there are 92 bolls in 47 quarters of the ſame meaſure, or 196 bolls nearly in 100 quarters.

By the help of theſe multipliers, the contents of veſſels may be found by the following rules.

1. To find the content of a cylindric veſſel; multiply the ſquare of the diameter of the baſe, by the height, both being expreſſed in inches; then, to find the content in wine-gallons, multiply the product by .0034; or to have it in ale-gallons, multiply by .0027851; or to have it in malt-buſhels, multiply it by .0003653; or again, to have the content in Scots pints, multiply the ſaid product by .0076; or if it be required in Scots wheat-firlots, multiply by .000358; or in barley-firlots, by .000245.

If the baſe of the veſſel be not a circ[illegible]t an ellipſe, multiply the product of the greateſt and [illegible] diameters, by the height, and then by the proper multiplier.

Let the diameter of the baſe be 48 inches, and the height 54 inches; the ſquare of 48, viz. 2304, multiplied by 54, gives 124416, the common product, which, multiplied by .0034, gives 423.0144 wine-gallons for the content; or if it be multiplied by .0027851, the content will be 346.511 ale-gallons.

2. When the veſſel may be conſidered as the fruſtum of a cone; from the ſquare of the ſum of the greateſt and leaſt diameters, ſubtract their product, and multiply ⅓ of the remainder by the length, and then by the multiplier for the meaſure.

Thus, if the length of ſuch a caſk be 40 inches, and the bung-diameter 32 inches, and the head-diameter 26 inches; their ſum is 58, from the ſquare of which, 3364, ſubtract their product

Fig 3

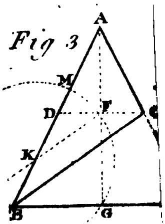
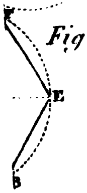


Fig 6



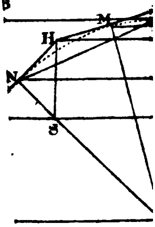


Fig 12

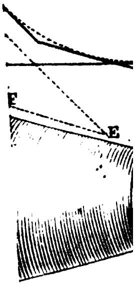
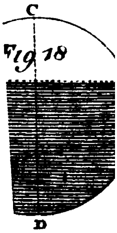


Fig 18



832, and the remainder is 2532; and ⅓ of it 844, multiplied by the length 40, gives 33760, which, multiplied by .0034, gives 114.7154, the content in wine-gallons.

3. When the veſſel is of the kind mentioned in Prob. 3. which is called a fruſtum of a parabolic conoid; take the diameter in the middle, between the greateſt and leaſt widths, and ſquare it; or add the ſquares of the greateſt and leaſt diameters, and take half of their ſum; then multiply by the length, and the multiplier.

4. If the veſſel be the fruſtum of a ſphere; from the ſquare of the greateſt diameter, ſubtract ⅓ of the ſquare of the diſtance between the greateſt and leaſt diameters, and multiply the remainder by the length, and by the multiplier.

5. If the veſſel be the fruſtum of a ſpheroid; to twice the ſquare of the greateſt diameter, add the ſquare of the leaſt, and multiply ⅓ of the ſum by the length, and the multiplier: but if the baſes be unequal, find a fourth proportional to the ſquare of the axis, the ſquare of conjugate axis of the ellipſe, and 1⅓ of the ſquare of the length, and add this fourth proportional to the ſquares of the greateſt and leaſt diameters, and multiply ½ the ſum by the length, and by the multiplier.

6. When the veſſel is the fruſtum of a hyperbolic conoid; find a fourth proportional to the ſquare of the axis, the ſquare of the conjugate axis, and 1⅓ of the ſquare of the length, and ſubtract it from the ſum of the ſquares of the greateſt and leaſt diameters; then multiply ½ of the remainder by the length, and by the multiplier.

7. When the veſſel is the fruſtum of a parabolic ſpindle; to twice the ſquare of the greateſt diameter, add the ſquare of the leaſt, and from the ſum ſubtract ⅖ of the ſquare of the difference of the ſame diameters, and multiply ⅓ of the remainder by the length, and by the multiplier for the meaſure required.

The moſt common forms of caſks contain leſs than what is given by the 4th, 5th, or 7th rule; but more than what is given by any of the others. The following method ſeems to agree beſt with common caſks, and is nearly a mean betwixt the 3d and 7th rules.

8. From 12 times the head-diameter, ſubtract 7 times the bung-diameter, and multiply the remainder by twice the bung-diameter, and ſubtract the product from the ſquare of 5 times the ſum of theſe diameters; then multiply the remainder by the length, and by .00034, and the product, divided by 9 for wine, or by 11 for ale, gives the content in gallons. The dimenſions being as in the 2d rule, the content by this rule would be 118½ wine-gallons, or 97 ale-gallons.

9. But the beſt way is to take a diameter in the middle, be-

tween the head and bung, and to add together the ſquares of the head and bung diameters, and of twice the middle diameter; for $\frac{1}{6}$ of the ſum reduces the caſk to a cylinder; that is, when multiplied by the length, and the proper multiplier, it gives the content, whatever be the form of the caſk.

10. Large veſſels are meaſured by diſtinguiſhing them into fruſtums, and taking the diameter at the middle of each fruſtum, with which the content of the fruſtum is found, as if it were a cylinder; and the contents of all the fruſtums added together give the content of the veſſel: after which, they form a table of the contents at each inch deep, by ſubtracting ſucceſſively the areas of the circles belonging to thoſe mean diameters, each as often as there are inches in the altitude of the fruſtum to which it belongs; and from this table it readily appears how much liquor is contained in the veſſel, by only knowing the depth of the liquor.

11. To meaſure round timber; find the mean circumference, or girth, and multiply its ſquare by .0795774715, which is the area, when the circumference is 1, the product is the content. Let the mean girth of a tree be 10.3 feet, and the length 24 feet, then the ſquare of 10.3, is 106.09, which, multiplied by 40, and by .0795775, gives 202.617 feet for the content. But the common way is to take $\frac{1}{4}$ of the girth, and to multiply its ſquare by the length for the content; according to this method, the tree, in this example, would be only 159.135 feet, which is 43.482 feet too little.

Square timber is meaſured by multiplying the mean breadth and thickneſs together, and the product by the length.

12. To find a ſhip's tonage; multiply the length of the keel by the breadth of the ſhip in the middle, and then multiply the product by the depth of the hold, and the product, divided by 94, gives the tonage; but neither this, nor any other rule that can be given, will ſerve for all ſorts of ſhips.

OF THE SLIDING RULE,

AND THE DIAGONAL ROD.

THE officers of the revenue generally find the contents of veſſels by the ſliding rule, or the diagonal rod. The lines on theſe inſtruments are divided into parts, proportional to the logarithms of numbers; that is, the diſtance between 1 and

10 being divided into 1000 equal parts, the number 2 is placed at 301 of theſe parts, and the number 3 at 477 of them; and ſo on; becauſe theſe are the logarithms of 2 and 3, when the logarithm of 10 is 1000. The diſtance between 1 and 10, on the line marked D, is three times as great as that between 1 and 10 on the line E, and twice as great as that on any of the other lines A, B, C, N, or MD; and therefore the numbers on E are the cubes of thoſe oppoſite to them on D, and the numbers on the other lines are the ſquares of thoſe on D. Any proportion may be wrought on the ſliding rule, by ſetting the firſt term on the rule oppoſite to the third on the ſlider; and then, oppoſite to the ſecond term on the rule, will be found the fourth term, or anſwer on the ſlider: Or it may be wrought on the rod, by extending the compaſſes from the firſt to the third term, and that extent, laid the ſame way from the ſecond term, will reach to the fourth term. The lines, marked SS, and SL, are uſed for ullaging ſtanding and lying caſks; ſet the length of a ſtanding caſk, or the bung-diameter of a lying caſk on N, oppoſite to 100 on the other lines, and oppoſite to the wet inches on N, take off the number on SS, if it be a ſtanding caſk, or on SL, if it be a lying caſk; then having placed the content of the caſk on A, oppoſite to 100 on B, look on B, for the number taken off, and oppoſite to it, on A, is the quantity of ale-gallons in the caſk. In like manner, on the rod, extend the compaſſes on the line of numbers, from the bung-diameter to the wet inches; and that extent will reach on SL, from 100 to a number, and the extent on the numbers from 1 to the content of the caſk, will reach from that number to the quantity of liquor in the caſk, in ale-gallons.

There are alſo lines on the rod, and on the inſide of the ſliders, for ſhewing the contents of cylinders, at one inch deep, in ale and wine gallons; and beſides theſe, on the ſliding rule there are lines marked ſpheroid or 1ſt variety, 2d variety, 3d variety, and conical variety, on the inſide of the ſlider N, for finding the mean diameters of caſks; that is, the number belonging to the variety, and oppoſite to the difference between the bung and head diameters, added to the head-diameter, gives the mean diameter, or the diameter of a cylinder of the ſame length and content with the caſk. Theſe varieties are the caſks mentioned in the 5th, 7th, 3d, and 2d rules; but the mean diameters found by the ſliding rule are not exact. The moſt remarkable lines on the rod, are thoſe called Diagonals; for if the rod be put into a caſk from the bung to the fartheſt edge of the head, the numbers on theſe lines, which are cut by the bung, will ſhew how many ale and wine gallons the caſk can hold; and for common caſks, this method ſeems to give the content more exactly than any rule that depends on their forms, and very nearly the ſame with the 9th rule, which in every caſe gives the true content.

Since on theſe inſtruments every thing is performed by proportions, the multipliers mentioned in the firſt rule muſt be converted into diviſors, by annexing cyphers to 1, and then dividing it by theſe multipliers, and the quotients will be diviſors, anſwering the ſame purpoſe; but in uſing the line D, the ſquare roots of theſe diviſors are uſed in the firſt rule, or the ſquare roots of 3 times the diviſors, in working any of the varieties, or of 6 times the ſame diviſors, in uſing the 9th rule; theſe are called Guage-points on D, and for cylinders, they are 18.95 for ale-gallons, and 17.15 for wine; and for any of the varieties, except the third, which is the ſame as for the cylinder, they are 32.8 for ale, and 29.7 for wine: and for the 9th rule, they are 46.4 for ale, and 42 for wine gallons.

For example, let the depth of a veſſel be 56 inches, and the mean diameter 42 inches; ſet 56 on C, oppoſite to 18.95 on D, then oppoſite to 42 on D, is 275 on C, which is the content in ale-gallons; or if 56 on C be ſet to 17.15 on D, then oppoſite to 42 on D, is 336 wine-gallons on C.

Again, let the length of a caſk of the conical variety be 40 inches, and the bung-diameter 32 inches, and the head 24 inches: firſt ſet 24 on C, to 24 on D, and oppoſite to 32 on C, is 27.7 on D, the ſquare root of the product of 32 and 24; therefore, ſet 40 on C, to 32.8 on D, and then oppoſite to 56, and 27.7 on D, are 116.5, and 28.5 on C, of which the difference is 88 ale-gallons, the content. Whenever the baſe of a veſſel has two diameters, the ſquare root of their product is to be found on D, and uſed inſtead of them, in the ſame manner as 27.7 in this example.

Again, let the length be 40 inches, the bung-diameter 32, the head 26, and the diameter in the middle between the head and bung 30.4 inches; and working as in the 9th rule, ſet 40 on C, oppoſite to 42 on D, and then oppoſite to 26, and 32, and 60.8 on D, there are found on C, 15.3, and 23.2, and 84.8, which, added together, give 122.3 wine-gallons for the content: and in the ſame manner, may any other of the above rules be wrought on the ſliding rule.

Some of theſe numbers are marked on the line D, and are called Guage-points, ſuch as W.G at 17.15, and A.G at 18.95, which are the guage-points for wine and ale gallons, in finding the contents of cylindrical veſſels; alſo M.R at 52.32, being the guage-point for malt-buſhels in the ſame veſſels, and M.S at 46.37, the guage-point for malt-buſhels, in finding the contents of ſquare or rectangular veſſels: and in the ſame manner, may any of the other guage-points be marked on the line D. And beſides the guage-points, there are ſeveral other numbers marked on the rule, ſuch as M.B at 2150.4, the inches in a malt-buſhel, A at 282, the inches in an ale-gallon; theſe are marked on the line A: and on the line B, there are marked W at 231, the inches in a wine-gallon, S.I at 707, and S.e at 886, which are the ſides of ſquares inſcribed, and equal to the circle of which the diameter is 1000, and C at 3141.6, the circumference of the ſame circle: and on the line C are marked OC at .0796, the area of a circle of which the circumference is 1, and Od at .7854, the area of a circle of which the diameter is 1. The uſe of theſe is obvious; for if the area of a circle be required of which the diameter is 8, place 1 on D, oppoſite to .7854, or Od on C, then, oppoſite to 8 on D, there will be found 50.3 on C, the area required.

The uſe of the line MD, is for guaging rectangular veſſels, or floors of malt: ſet the length on B, oppoſite to the breadth on MD, and then oppoſite to the depth on A, there will be found the content in malt-buſhels on B. Thus, if the length of a floor of malt be 270 inches, its breadth 56 inches, and its mean depth 5 inches; ſet 270 on B, oppoſite to 56 on MD, counted towards the left hand, then oppoſite to 5 on A, there is found 35 on B, the number of buſhels in the floor.

By the line E, the contents of ſimilar veſſels may be found from one another, and the dimenſions of ſimilar veſſels may be found from their contents. Suppoſe the depth of a veſſel containing 100 gallons to be 40 inches; what is the content of a ſimilar veſſel, of which the depth is 36 inches? ſet 40 on D, againſt 100 on E, then, oppoſite to 36 on D, there will be found 72.9 on E, which is the content. Again, let the depth of a veſſel be 30 inches, and that it is required to find the depth of a ſimilar veſſel that ſhall contain twice as much; ſet 30 on D, to 1 on E, and againſt 2 on E, there will be found 37.8 on D, which is the depth required. Suppoſe, again, that the diagonal; a caſk containing a hogſhead, is 31 inches, and that it is required to find the diagonal of a ſimilar caſk that ſhall hold a puncheon; ſet 31 on D, to 63 gallons on E, and againſt 84 gallons on E, there will be found 34.1 inches on D, which is the diagonal required.

It is evident, that the ſame rules will give the contents of veſſels in the meaſures of foreign countries, if proper multipliers or guage-points be found for theſe meaſures; and theſe multipliers and guage-points are found from the number of inches in any meaſure exactly in the ſame manner with thoſe for Engliſh meaſures. For example, the ſextier of Paris contains 384 cubical Paris inches; therefore, if .785398 be divided by

384, the quotient .0020453 is a multiplier for giving the contents in ſextiers; or if 384 be divided by .785398, the quotient 488.924, will be a diviſor for the ſame purpoſe, and the ſquare root 22.1 of this diviſor will give the guage-point for cylinders to be uſed on the line D: Theſe numbers ſuppoſe that the veſſel is guaged in Paris inches, but if the veſſel be guaged in Engliſh inches, the ſextier contains 464.8 ſuch inches; and therefore the multiplier in this caſe will be .00169, and the guage-point on D will be 24.3. In like manner, there being 1040 cubical Engliſh inches in a ſtechan of Amſterdam, or in an almuda of Portugal, numbers may be found for finding the contents in the meaſures of theſe countries; and ſo on.

When a caſk, not full, is lying with its axis parallel to the horizon, it was ſhown in Prob. 9. of this Part, how to find the content of the full or empty part; but it required the area of the ſegment of a circle to be found, which being tedious, is uſually done by means of a table of circular ſegments, of which the following is a ſpecimen.

A TABLE

Of the Areas of Segments.

V. S.	Segm.	V. S.	Segm.	V. S.	Segm.	V. S.	Segm.	V. S.	Segm.
01	00133	11	04701	21	11990	31	20738	41	30319
02	00375	12	05339	22	12811	32	21667	42	31304
03	00687	13	06000	23	13647	33	22603	43	32293
04	01054	14	06683	24	14494	34	23547	44	33284
05	01468	15	07387	25	15355	35	24498	45	34278
06	01924	16	08111	26	16226	36	25455	46	35274
07	02417	17	08854	27	17109	37	26418	47	36272
08	02944	18	09613	28	18002	38	27386	48	37270
09	03501	19	10390	29	18905	39	28359	49	38270
10	04088	20	11182	30	19817	40	29337	50	39270

To ullage a lying caſk by the table of ſegments, divide the wet inches by the bung-diameter, and find the quotient in the table, in one of the columns marked V. S. and the area of the ſegment neareſt to it on the right ſide, is to be taken out of the table, and multiplied by the whole content of the caſk; divide the product by 8, and from the quotient cut off four figures for a decimal, the reſt of the figures ſhew the quantity of ale-gallons in the caſk.

Suppoſe the content of a caſk to be 92 gallons, the bung-diameter 32 inches, and the wet inches 8; divide 8.00 by 32, the quotient is .25, oppoſite to which in the table is found 15355, which, multiplied by 92, and four figures cut off from the product, gives 141.3660; and this, divided by 8, gives 18 ale-gallons nearly for the quantity of liquor in it.

If the caſk be above half full, divide the dry inches inſtead of the wet inches, and thus find the content of the empty part, which, ſubtracted from the whole content, gives the quantity of liquor in it.

TABLE

Of Foreign Meaſures expreſſed in Engliſh Inches.

		Inches.		Inches.
Paris	foot	12.792	aune	46.680
Lyons		13.458	aune cane	47.604 77.220
Leyden		12.396	aune	27.120
Amſterdam		11.304		27.228
Antwerp		11.352		27.276
Bruſſels		10.828		27.260
Hamburgh		11.376		22.860
Lubeck		11.448		22.896
Denmark		12.504		25.008
Sweden		11.733		23.466
Dantzick		11.328		22.836
Riga		10.986		21.972
Ruſſia	arſheen	9.090		
Rome	pal. merc. pal. arch.	9.791 8.779	brace mer. brace arch. cane.	34.270 30.730 81.900
Venice	foot	13.944	brace	26.460
Leghorn	palm	9.182	brace	22.956
Genoa	palm	9.960	cane	88.290
Naples	palm	10.332	brace cane	25.200 82.560
Madrid	palm foot	9.012 12.012	vara	36.040
Toledo	foot	10.788	vara	32.220
Gibraltar		11.039	vara	33.120
Liſbon	cavido	20.122	vara	33.000
Conſtantinople	ſh. pike	25.576	gr. pike	27.920
Perſia			ariſh	38.364
Old Greek	foot	12.087	cubit	18.132
Old Roman	foot	11.604	cubit	17.496
Scripture	palm	3.648	cubit	21.888

FINIS.